3

Recall: for $u_{tt} - c^2 u_{xx} = 0$

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t(\xi,t))^2 + c^2(u_x(\xi,t))^2 \, d\xi$$

We showed: $\frac{d}{dt}E(t) = 0$

$$\Rightarrow E(t) = E(0)$$

Well posedness of wave

- Existence – d'Alembert
- Uniqueness: Suppose u_1, u_2 both solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \Rightarrow \quad \begin{matrix} z_{tt} - c^2 z_{xx} = 0 \\ z(x,0) = 0 \\ z_t(x,0) = 0 \end{matrix}$$

Then apply energy identity

$$E_z(t) = E_z(0) = 0$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{\infty} z_t^2 + c^2 z_x^2 \, d\xi = 0 \quad \forall t$$

$$z_t^2, z_x^2 \geq 0$$

$$\Rightarrow z_t = z_x \equiv 0 \quad \forall t$$

$$\Rightarrow z \equiv 0 \Rightarrow u_1 = u_2 \quad \forall t$$

General uniqueness strategy

Continuous dependence

suppose different initial data

$$\begin{cases} u^i_{tt} - c^2 u^i_{xx} = 0 \\ u^i(x,0) = \varphi^i(x) \\ u^i_t(x,0) = \psi^i(x) \end{cases} \Rightarrow \qquad z = u^1 - u^2$$

Again

$$E_z(t) = E_z(0) = \frac{1}{2}\int_{-\infty}^{\infty} c^2(\varphi^1-\varphi^2)_x^2 + (\psi^1-\psi^2)^2 \, d\xi$$

If (φ^1, ψ^1) "close" to (φ^2, ψ^2), then (u^1, u^1_t) "close" to (u^2, u^2_t) in the sense of the energy of their difference (square integral sense).

Lower order equations

damped wave $\qquad u_{tt} + d u_t - c^2 u_{xx} = 0$

$$E(t) + d\int_0^t \int_{-\infty}^{\infty} (u_t)^2 \, d\xi \, d\tau = E(0)$$

method

Klein Gordon $\qquad u_{tt} - c^2 u_{xx} + m^2 u = 0$

$$\hat{E}(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 + c^2(u_x)^2 + m^2 u^2 \, d\xi$$

$$\hat{E}(t) = \hat{E}(0).$$

5

General lower order

$$u_{tt} - c^2 u + a_1 u_x + a_2 u_t + a_3 u = F(x,t)$$

1. Reduce to Klein-Gordon via $u = v\, e^{\alpha x - \beta t}$ HW

2. Solving K-G below + HW

3. $F(x,t) \neq 0$, Duhamel (in memory) HW

Recall dispersion relations:

Assume $u(x,t) = A(\,)\, e^{i(kx - \omega t)}$ — wave train sol^n

ω / k wave speed

$\partial_t \mapsto -i\omega \qquad \partial_x \mapsto ik$

wave $\quad u_{tt} = c^2 u_{xx} \qquad \dfrac{\omega}{k} = \pm c$

neither dispers. or dissip.

damped wave

$$u_{tt} + d u_t = c^2 u_{xx}$$

$$-\omega^2 - i\omega + c^2 k^2 = 0$$

$$\omega = \frac{-i \pm \sqrt{-1 + 4c^2k^2}}{2} = \frac{-i}{2} + \alpha$$

$$u = A e^{i(kx - \omega t)} = A e^{ikx} e^{-i\omega t} = A e^{ikx} e^{-t/2} e^{-i\alpha/2}$$

What happens? to wave trains?

Klein-Gordon

$u_{tt} = c^2 u_{xx} - m^2 u$

$-\omega^2 + (ck)^2 + m^2 = 0$

$\omega = \pm\sqrt{(ck)^2 + m^2}$

So $Ae^{i(kx-\omega t)}$ a sol^n when

$e^{i(kx-\omega t)} = e^{ik(x - \omega/k\, t)}$

$\frac{\omega}{k} = \pm\frac{\sqrt{(ck)^2+m^2}}{\sqrt{k^2}} = \pm\sqrt{c^2 + \left(\frac{m}{k}\right)^2}$

Wave speed changes as wave length changes

Family of solutions of the form

$u(x,t;k) = A_+(k)e^{i(kx-\omega_+(k)t)} + A_-(k)e^{i(kx-\omega_-(k)t)}$

$u(x,t) = \int_{-\infty}^{\infty} A_+(k)e^{i(kx-\omega(k)t)} + \_\_\_$

also a solution!

Wave Equation in $n=1,2,3$

$n=1$ $\quad u_{tt} - c^2 u_{xx} = 0 \qquad u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x)$

$$u(x,t) = \frac{1}{2}\left\{ \varphi(x+ct) + \varphi(x-ct) \right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$n=2$ $\quad u_{tt} - c^2(u_{xx}+u_{yy}) = 0 \qquad u(x,y,0) = \varphi(x,y)$

$u_t(x,y,0) = \psi(x,y)$

$$D = D(x,y,t) = \left\{ (s_1,s_2) : \sqrt{(x-s_1)^2+(y-s_2)^2} \le ct \right\}$$

$$u(x,t) = \frac{1}{2\pi c}\iint_D \frac{\psi(s_1,s_2)\,ds_1\,ds_2}{\left((ct)^2-(x-s_1)^2-(y-s_2)^2\right)^{1/2}} + \frac{\partial}{\partial t}\left\{ \iint_D \frac{\varphi(s_1,s_2)\,ds_1\,ds_2}{\left((ct)^2-(x-s_1)^2-(y-s_2)^2\right)^{1/2}} \right\}$$

$n=3$ $\quad u_{tt} - c^2\Delta u = 0 \qquad u(\vec{x},t) = \varphi(\vec{x}),\ u_t(\vec{x},0) = \psi(\vec{x})$

$$S = S(\vec{x},t) = \{ \vec{x}' : \|\vec{x}'-\vec{x}\| = ct \}$$

$$u(\vec{x},t) = \frac{1}{4\pi c^2 t}\iint_S \psi(\vec{x}')\,d\vec{x}' + \frac{\partial}{\partial t}\left\{ \frac{1}{4\pi c^2 t}\iint_S \varphi(\vec{x}')\,d\vec{x}' \right.$$

Dependencies?

Note

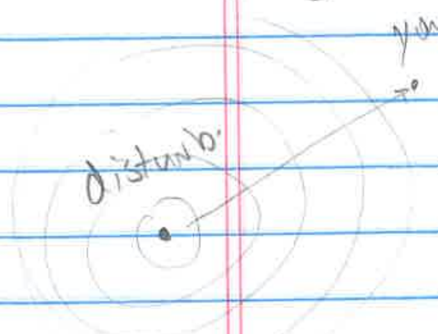
In 1-D solution can depend on info in $[x-ct, x+ct]$ (w/ 4 data)

In 2-D solution depends on info in $D(\vec{x}, t)$

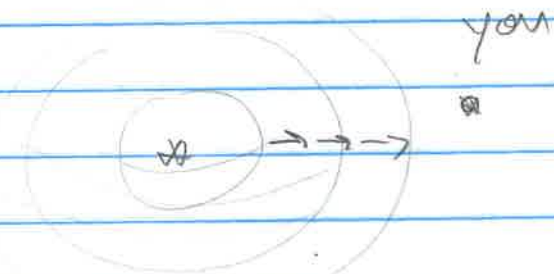
Key

In 3-D solution depends on info on $S(\vec{x}, t)$

2-D



3-D



Sharp signal propagation

Huygen's principle

In 3-D (and any dim. $n \geq 3$, n odd) signal propagate at exactly speed $\underline{c}$.