Wave Equation Cauchy Problem ①

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \varphi(x) & x \in \mathbb{R}, t = 0 \\ u_t(x,0) = \psi(x) & x \in \mathbb{R}, t = 0 \end{cases}$$

WEQ is a linear, homogeneous, second order, hyperbolic equation.

The d'Alembert Sol^{n}

$$u(x,t) = \frac{1}{2}\{\varphi(x+ct) + \varphi(x-ct)\} \quad (1)$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \quad (2)$$

This is the unique solution among those that satisfy conservation of energy.

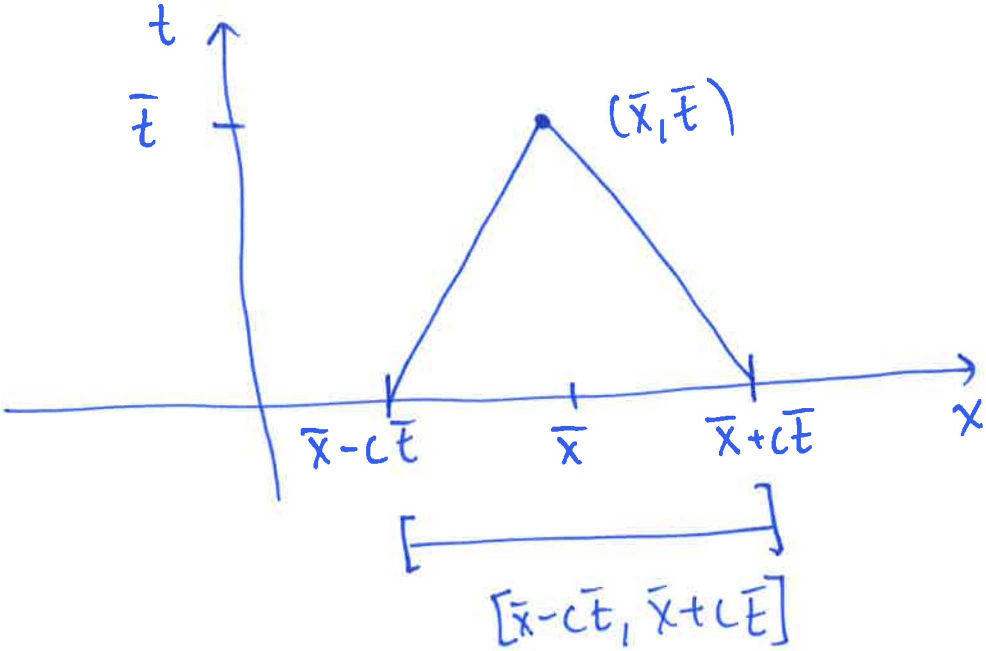
By <u>superposition</u>, (1) is the result of the effect of the initial displacement; (2) is the result of the effect of the initial velocity.

Reconstructing the solution from the data

d'Alembert can be viewed as the output fro the input (φ, ψ):

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \longmapsto u(x,t) = \frac{1}{2}\{\varphi(x+ct) - \varphi(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

If $(\bar{x}, \bar{t})$ is a pt in space time, let's think about what info is needed to "reconstruct" $u(\bar{x}, \bar{t})$:



To "get" $u(\bar{x}, \bar{t})$, we need to know the value of $\varphi(\bar{x}+c\bar{t})$ and $\varphi(\bar{x}-c\bar{t})$ as well as $\psi(s)$ for $s \in [\bar{x}-c\bar{t}, \bar{x}+c\bar{t}]$.

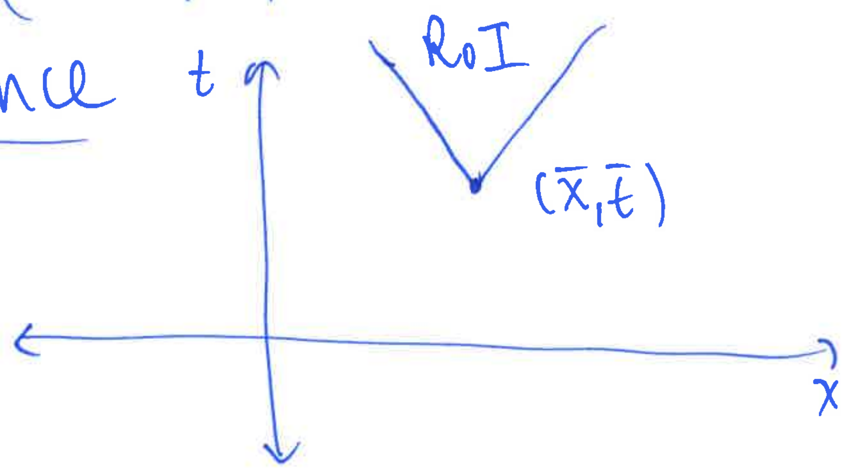
This creates what is called the domain of dependence for the solution at $(\bar{x},\bar{t})$:

sol^n at $(\bar{x},\bar{t})$ depends on the values of the initial data on $[\bar{x}-c\bar{t}, \bar{x}+c\bar{t}]$.

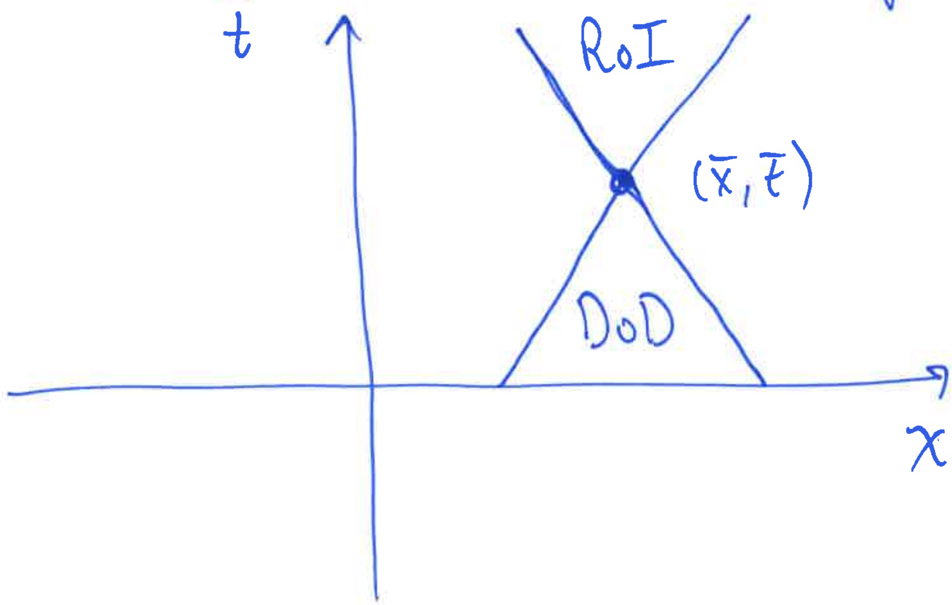
(Specifically, we need φ at the end points + ψ on the interior of the interval.)

Outside of $[\bar{x}-c\bar{t}, \bar{x}+c\bar{t}]$, the values of φ + ψ do not affect the solution at the pt (x,t) in space-time.

We can think of the effect of the pt $(\bar{x},\bar{t})$ into the "future". The value of $u(x,t)$ (at $(\bar{x},\bar{t})$) affects the region of influence



These are "dual" regions that, when taken together, form a <u>light cone</u>.



lines of slope $\pm\frac{1}{c}$

The RoI for $(\bar{x},\bar{t})$ is the set of pts in spacetime whose values depend on the quantity $u(\bar{x},\bar{t})$ (and its derivatives).

The DoD for $(\bar{x},\bar{t})$ is the set of points in spacetime whose data values determine the solution value at $(\bar{x},\bar{t})$.

If $(\tilde{x},\tilde{t})$ is outside the light cone for $(\bar{x},\bar{t})$, then $u(\tilde{x},\tilde{t})$ <u>has no effect on</u> + <u>is not affected by</u> $u(\bar{x},\bar{t})$.