Wave equation on $\mathbb{R}$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

d'Alembert:

$$u(x,t) = \frac{1}{2}\{\varphi(x+ct) + \varphi(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Ex 1 $\varphi(x) = \sin(x)$, $\psi(x) \equiv 0$

$$u(x,t) = \frac{1}{2}\{\sin(x+ct) + \sin(x-ct)\}$$

standing wave

$$= \frac{1}{2}\big(\sin x \cos(ct) - \cos(x)\sin(ct) + \sin(x)\cos(ct) + \cos(x)\sin(ct)\big)$$

$$= \sin(x)\cos(ct)$$

Ex 2 $\varphi(x) = 0$, $\psi(x) = \frac{1}{1+x^2}$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+s^2}\,ds$$

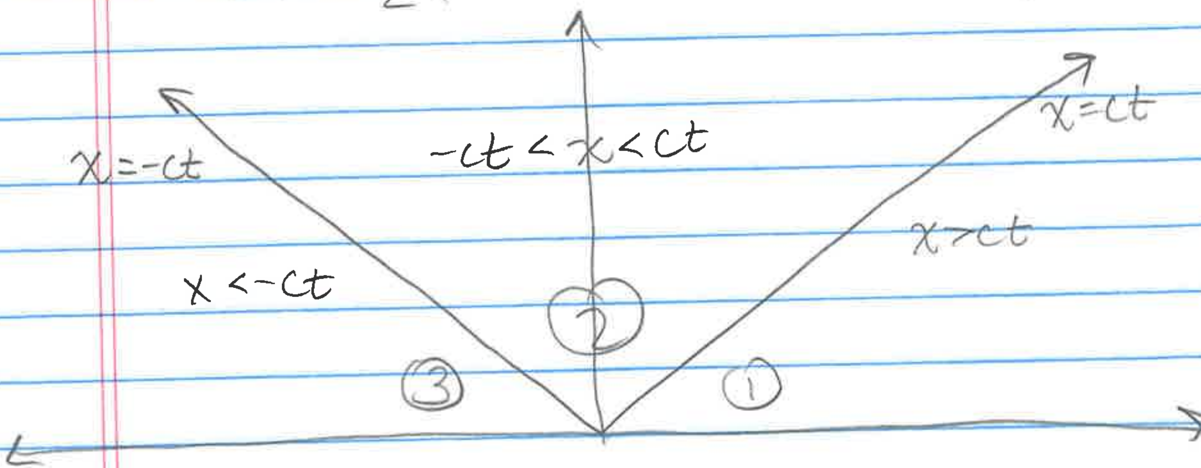
$$= \frac{1}{2c}\big(\arctan(x+ct) - \arctan(x-ct)\big)$$

Picture

Ex 3 $\varphi(x) = H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

$\psi(x) \equiv 0$

$u(x,t) = \frac{1}{2}\left(H(x-ct) + H(x-ct) \right)$



$H(x-ct) = \begin{cases} 0 & x-ct < 0 \leftrightarrow x < ct \\ 1 & x-ct \geq 0 \leftrightarrow x \geq ct \end{cases}$

$H(x+ct) = \begin{cases} 0 & x+ct < 0 \leftrightarrow x < -ct \\ 1 & x+ct \geq 0 \leftrightarrow x \geq -ct \end{cases}$

$u(x,t) = \begin{cases} 0 & \text{in Region } 3 \\ \frac{1+0}{2} = \frac{1}{2} & \text{in Region } 2 \\ \frac{1+1}{2} = 1 & \text{in Region } 1 \end{cases}$

Comments: Jump discontinuity in data $\rightarrow$ sol^{n} discont. propagate along characteristics

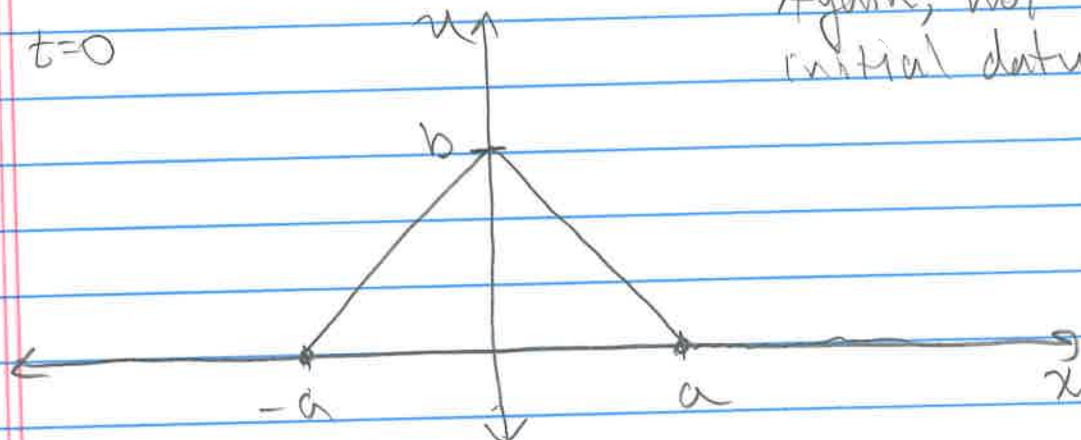
Ex 4 (Three finger pluck)

$$\varphi(x) = \begin{cases} b - \frac{b|x|}{a} & |x| < a \\ 0 & |x| \geq a \end{cases}$$

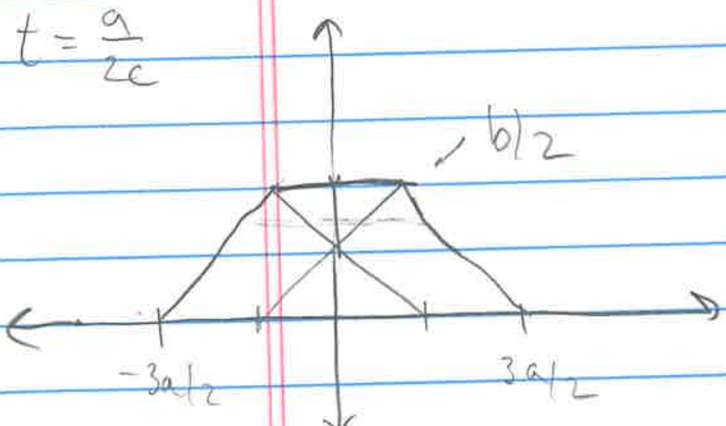
$\psi(x) = 0$

Again, not "smooth" initial datum

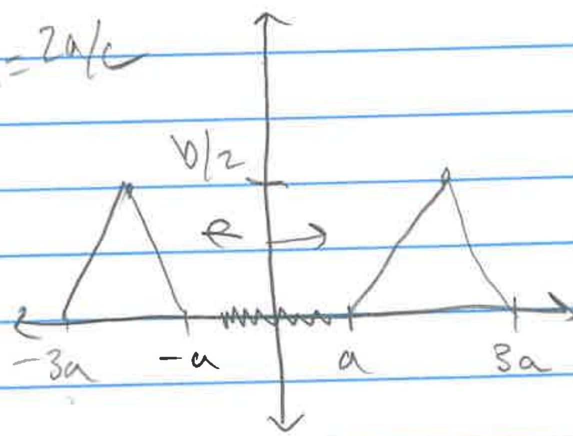
$t=0$



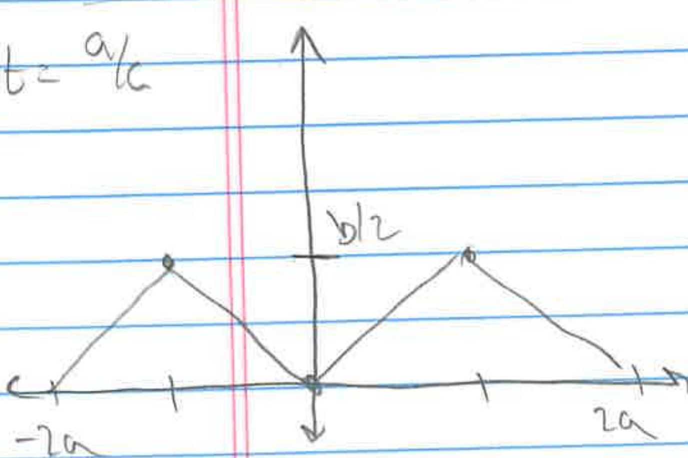
$t = \frac{a}{2c}$

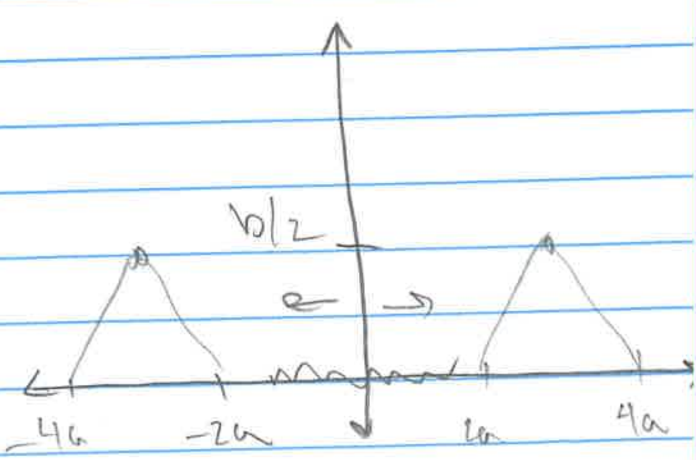


$t = 2a/c$



$t = a/c$





Comments

Again: singularities < along charact. / φ cont, φ_x not / true $\forall$ $\underline{x,t}$

Energy is $\underline{\text{conserved}}$!

$$\rho u_{tt} - T u_{xx} = 0$$

$$E(t) = \frac{\rho}{2}\int_{-\infty}^{\infty} (u_t(\xi,t))^2 \, d\xi + \frac{T}{2}\int_{-\infty}^{\infty} (u_x(\xi,t))^2 \, d\xi$$

We care about solutions (u, u_t) so that $E(t) < +\infty \quad \forall t$

(e.g. if solutions ① decay rapidly as $|x| \to \infty$
② are compctly supported

$u_{tt} - c^2 u_{xx} = 0$

Compute $\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} (u_t)^2 \, d\xi\right)$

$$\left(= \int_{-\infty}^{\infty} u_t \, u_{tt} \right)$$

$$\frac{d}{dt}\left(\frac{c^2}{2}\int_{-\infty}^{\infty} (u_x)^2 \, d\xi\right) = c^2 \int_{-\infty}^{\infty} u_x u_{xt} \, d\xi$$

$$= -c^2 \int_{-\infty}^{\infty} u_{xx} u_t \, d\xi$$

assume away $\longrightarrow$ $+ \cancel{u_x u_t \Big|_{-\infty}^{\infty}} \; 0$

$u_{tt} - c^2 u_{xx} = 0 \quad / \; u_t \; / \int_{-\infty}^{\infty}$

$$\frac{d}{dt}(E(t)) = 0 \implies \boxed{E(t) = E(0)}$$