Wave Equation on $\mathbb{R}$ w/ wave speed c

$$u_{tt} - c^2 u_{xx} = 0 \qquad x \in \mathbb{R},\ t > 0$$

Cauchy / Initial Data

$$u(x,0) = \varphi(x) \qquad u_t(x,0) = \psi(x)$$

So the wave equation IVP is

$$(*) \quad \begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

The d'Alembert solution to $(*)$ is:

$$(\mathrm{D}) \quad u(x,t) = \frac{1}{2}\left\{ \varphi(x+ct) + \varphi(x-ct) \right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)\, d\tau$$

Note: The general solution to the wave equation is

$$u(x,t) = f(x+ct) + g(x-ct)$$

for arbitrary smooth f, g.

In contrast, the WEIVP has only the d'Alembert (unique!) solution.

Let us verify that (D) solves (*).

First note:

$$u(x,0) = \frac{1}{2}\{\varphi(x+0c) + \varphi(x-c0)\} + \frac{1}{2c}\int_{x-0}^{x+0} \psi(\tau)\,d\tau$$

$$= \frac{2}{2}\varphi(x) + 0 = \varphi(x) \checkmark$$

$$u_t(x,t) = \frac{1}{2}\{c\varphi'(x+ct) - c\varphi'(x-ct)\} + \frac{1}{2c}\{\psi(x+ct)\cdot[c] - \psi(x-ct)[-c]\}$$

$$= \frac{c}{2}\{\varphi'(x+ct) - \varphi'(x-ct)\} + \frac{1}{2}\{\psi(x+ct) + \psi(x-ct)\}$$

So $u_t(x,0) = \frac{c}{2}\{\varphi'(x) - \varphi'(x)\}$

$+ \frac{1}{2}\{2\psi(x)\}$

$= \psi(x)$ ✓

Now let's compute $u_{tt} + u_{xx}$:

$$u_{tt} = \frac{c^2}{2}\{\varphi''(x+ct) + \varphi''(x-ct)\} + \frac{1}{2}\{c\psi'(x+ct) - c\psi'(x-ct)\}$$

$$u_x = \frac{1}{2}\{\varphi'(x+ct) + \varphi'(x-ct)\} + \frac{1}{2c}\{\psi(x+ct) - \psi(x-ct)\}$$

$$u_{xx} = \frac{1}{2}\{\varphi''(x+ct) + \varphi''(x-ct)\} + \frac{1}{2c}\{\psi'(x+ct) - \psi'(x-ct)\}$$

Check: $u_{tt} = -c^2 u_{xx}$ ✓

Yes so (D) solves the WEQ + both initial conditions.

1

$$u_{xx} + u_{xt} - 20u_{tt} = 0$$

$$u_{tt} - \frac{1}{20}u_{xt} - \frac{1}{20}u_{xx} = 0$$

$$\left(\partial_t - \frac{\partial_x}{4}\right)\left(\partial_t + \frac{\partial_x}{5}\right)u = 0$$

$$\xi = x + t/4 \qquad \eta = x - t/5$$

Change variables + solve:

$$u(x,t) = f(x + t/4) + g(x - t/5)$$

Impose ICs:

$$\varphi = u(x,0) = f(x) + g(x) \implies \varphi'(x) = f'(x) + g'(x)$$

$$\psi = u_t(x,0) = \frac{1}{4}f'(x) - \frac{1}{5}g'(x)$$

$$-\frac{1}{4}\varphi'(x) + \psi = -\frac{9}{20}g' \implies \boxed{g'(x) = \frac{5}{9}\varphi'(x) - \frac{20}{9}\psi}$$

$$f'(x) = \varphi'(x) - g'(x) \implies f'(x) = \frac{4}{9}\varphi'(x) + \frac{20}{9}\psi$$

$$u(x,t) = \frac{4}{9}\varphi(x + t/4) + \frac{5}{9}\varphi(x - t/5) + \frac{20}{9}\int_{x-t/5}^{x+t/4}\psi$$

$$u(x,0) = \frac{4}{9}\varphi(x) + 5/9\,\varphi(x) + 0$$ ✓

Verify: PDE

$$u_x = \frac{4}{9}\varphi'(x+t/4) + \frac{5}{9}\varphi'(x-t/5) + \frac{20}{9}\left[\psi(x+t/4) - \psi(x-t/5)\right]$$

$$u_t = -\frac{1}{9}\varphi'(x+t/4) + \frac{1}{9}\varphi'(x-t/5) + \frac{5}{9}\psi(x+t/4) + \frac{4}{9}\psi(x-t/5)$$

$$u_t(x,0) = 0 + \frac{4}{9}\psi(x) + \frac{5}{9}\psi(x) = \psi(x) \checkmark$$

$$1 \quad u_{xx} = \frac{4}{9}\varphi''(x+t/4) + \frac{5}{9}\varphi''(x-t/5) + \frac{20}{9}\psi'(x+t/4) + \frac{20}{9}\psi'(x-t/5)$$

$$20 \quad u_{tt} = \frac{\varphi''(x+t/4)}{36} + \frac{\varphi''(x-t/5)}{45} + \frac{5}{36}\psi'(x+t/4) - \frac{4}{45}\psi'(x-t/5)$$

$$1 \quad u_{xt} = -\frac{1}{9}\varphi''(x+t/4) + \frac{1}{9}\varphi''(x-t/5) + \frac{5}{9}\psi'(x+t/4) - \frac{4}{45}\psi'(x-t/5)$$

Works

3

$$u_{xx} + u_{xt} - 20u_{tt} = 0$$

$$\left[\partial_x^2 + \partial_x\partial_t - 20\partial_t^2\right] u = 0$$

$$(\partial_x - 4\partial_t)(\partial_x + 5\partial_t)u = 0$$

$$\xi = t + 4x \qquad \eta = t - 5x$$

$$u(x,t) = f(t+4x) + g(t-5x)$$

$$u_x = 4f' - 5g' \qquad u_t = f' + g'$$

$$u_{xx} = 16f'' + 25g' \qquad u_{tt} = f'' + g'' \qquad u_{xt} = 4f'' - 5g''$$

Still solves.

Algebra for imposing IC is quite awful here.