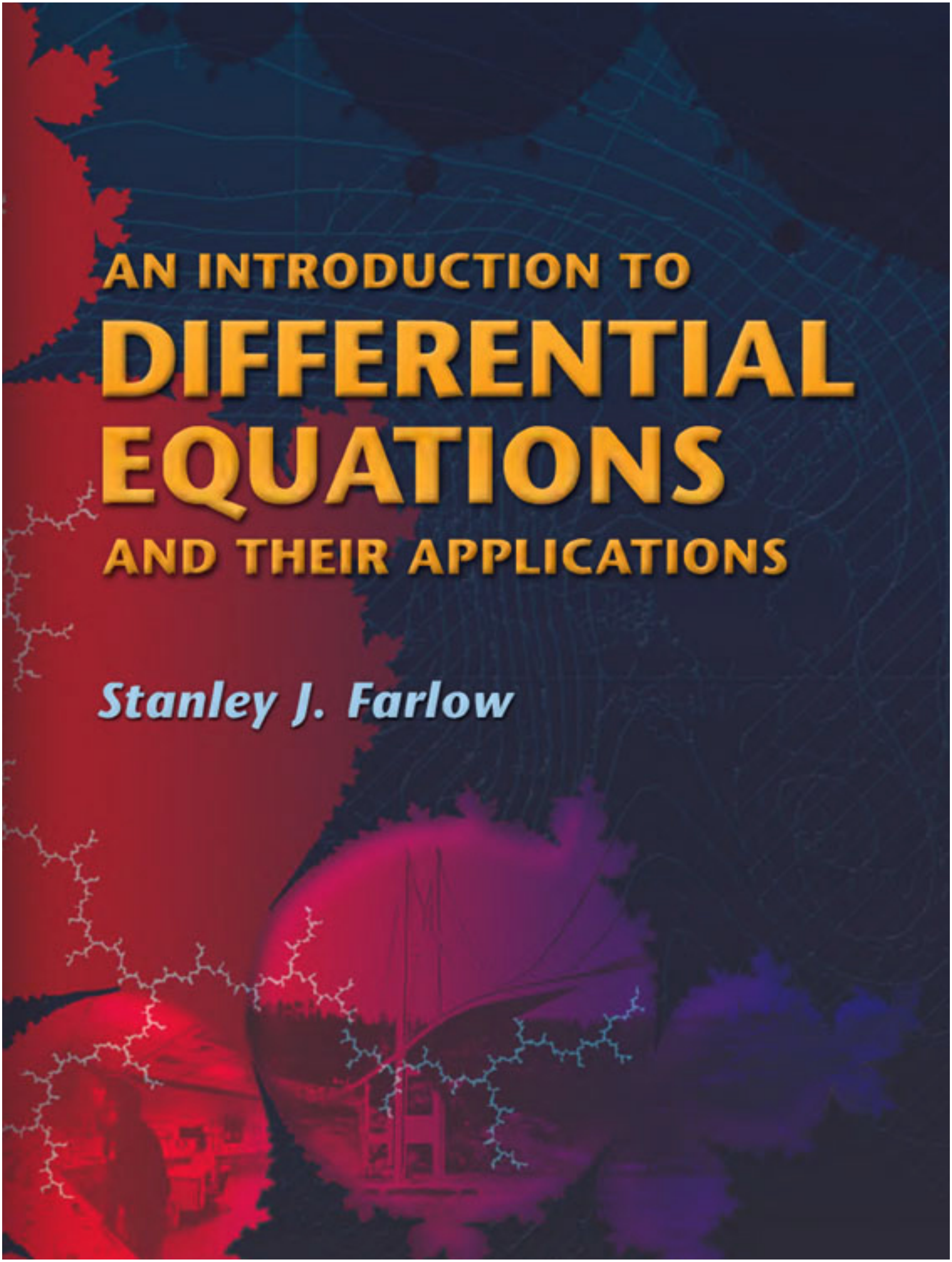
AN INTRODUCTION TO
DIFFERENTIAL
EQUATIONS
AND THEIR APPLICATIONS
Stanley J. Farlow

TABLE OF INVERSE LAPLACE TRANSFORMS

	$F(s)$	$f(t) = \mathfrak{L}\{F(s)\}$
1.	$\dfrac{1}{(s-a)(s-b)}$	$\dfrac{1}{a-b}(e^{at} - e^{bt})$
2.	$\dfrac{s}{(s-a)(s-b)}$	$\dfrac{1}{a-b}(ae^{at} - be^{bt})$
3.	$\dfrac{1}{s(s^2+a^2)}$	$\dfrac{1}{a^2}(1 - \cos at)$
4.	$\dfrac{1}{s^2(s^2+a^2)}$	$\dfrac{1}{a^3}(at - \sin at)$
5.	$\dfrac{1}{(s^2+a^2)^2}$	$\dfrac{1}{2a^3}(\sin at - at\cos at)$
6.	$\dfrac{s}{(s^2+a^2)^2}$	$\dfrac{t}{2a}\sin at$
7.	$\dfrac{s^2}{(s^2+a^2)^2}$	$\dfrac{1}{2a}(\sin at + at\cos at)$
8.	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
9.	$\dfrac{1}{(s-a)^2+b^2}$	$\dfrac{1}{b}e^{at}\sin bt$
10.	$\dfrac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$

PROPERTIES OF THE LAPLACE TRANSFORM:

$$F(s) = \mathfrak{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

1. $\mathfrak{L}\{f+g\} = \mathfrak{L}\{f\} + \mathfrak{L}\{g\}$
2. $\mathfrak{L}\{cf\} = c\mathfrak{L}\{f\}$
3. $\mathfrak{L}\{f'\} = s\mathfrak{L}\{f\} - f(0)$
4. $\mathfrak{L}\{f''\} = s^2\mathfrak{L}\{f\} - sf(0) - f'(0)$
5. $\mathfrak{L}\{f^{(n)}\} = s^n\mathfrak{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
6. $\mathfrak{L}\{e^{at}f(t)\} = F(s-a)$
7. $\mathfrak{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}F(s)$

8. $\mathfrak{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

9. $\mathfrak{L}\{f * g\} = \mathfrak{L}\{f\}\,\mathfrak{L}\{g\}$

10. $\mathfrak{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$

11. $\mathfrak{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\xi)\,d\xi$

12. $\lim_{s\to\infty} sF(s) = f(0)$

13. $\lim_{s\to 0} sF(s) = f(\infty)$

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University of Maine

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ABOUT THE AUTHOR

Stanley J. Farlow has academic degrees from Iowa State University, the University of Iowa, and Oregon State University. He is a former Lieutenant Commander and Public Health Service Fellow who served for several years as a computer analyst at the National Institutes of Health in Bethesda, Maryland. In 1968 he joined the faculty of the University of Maine, where he is currently Professor of Mathematics. He is also the author of *Partial Differential Equations for Scientists and Engineers*, (currently being published by Dover Publications, Inc.), *Finite Mathematics* (McGraw-Hill, 1988, 1994), *Applied Mathematics* (McGraw-Hill, 1988), *Introduction to Calculus* (McGraw-Hill, 1990), and *Calculus and Its Applications* (McGraw-Hill, 1990). He has also edited *The GMDH Method: Self-Organizing Methods in Modeling* (Marcel Dekker, 1984).

To Dorothy and Susan

CONTENTS

PREFACE

An Introduction to Differential Equations and Their Applications is intended for use in a beginning one-semester course in differential equations. It is designed for students in pure and applied mathematics who have a working knowledge of algebra, trigonometry, and elementary calculus. The main feature of this book lies in its exposition. The explanations of ideas and concepts are given fully and simply in a language that is direct and almost conversational in tone. I hope I have written a text in differential equations that is more easily read than most, and that both your task and that of your students will be helped.

Perhaps in no other college mathematics course is the interaction between mathematics and the physical sciences more evident than in differential equations, and for that reason I have tried to exploit the reader's physical and geometric intuition. At one extreme, it is possible to approach the subject on a highly rigorous "lemma-theoremcorollary" level, which, for a course like differential equations, squeezes out the lifeblood of the subject, leaving the student with very little understanding of how differential equations interact with the real world. At the other extreme, it is possible to wave away all the mathematical subtleties until neither the student nor the instructor knows what's going on. The goal of this book is to balance mathematical rigor with intuitive thinking.

FEATURES OF THE BOOK

Chaotic Dynamical Systems

This book covers the standard material taught in beginning differential equations

courses, with the exception of Chapters 7 and 8, where I have included optional sections relating to chaotic dynamical systems. The period-doubling phenomenon of the logistic equation is introduced in Section 7.5 and Julia sets and the Mandelbrot set are introduced in Section 7.6. Then, in Section 8.4, the chaotic behavior of certain nonlinear differential equations is summarized, and the Poincare section and strange attractors are defined and discussed.

Problem Sets

One of the most important aspects of any mathematics text is the problem sets. The problems in this book have been accumulated over 25 years of teaching differential equations and have been written in a style that, I hope, will pique the student's interest.

Because not all material can or should be included in a beginning textbook, some problems are placed within the problem sets that serve to introduce additional new topics. Often a brief paragraph is added to define relevant terms. These problems can be used to provide extra material for special students or to introduce new material the instructor may wish to discuss. Throughout the book, I have included numerous computational problems that will allow the students to use computer software, such as DERIVE, MATHEMATICA, MATHCAD, MAPLE, MACYSMA, PHASER, and CONVERGE.

Writing and Mathematics

In recent years I have joined the “Writing Across the Curriculum” crusade that is sweeping U.S. colleges and universities and, for my own part, have required my students to keep a scholarly journal. Each student spends five minutes at the end of each lecture writing and outlining what he or she does or doesn't understand. The idea, which is the foundation of the “Writing Across the Curriculum” program, is to learn through writing. At the end of the problem set in Section 1.1, the details for keeping a journal are outlined. Thereafter, the last problem in each problem set suggests a journal entry.

Historical Notes

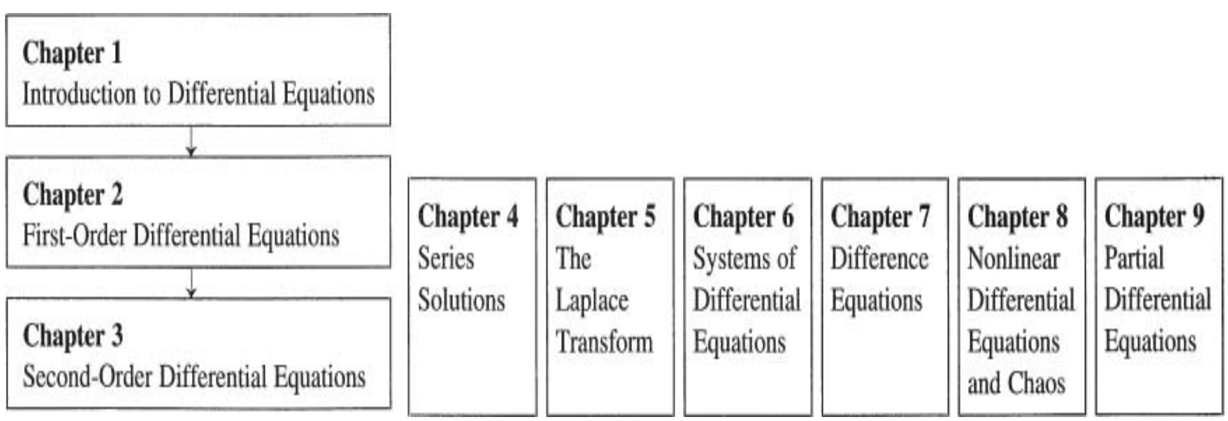
An attempt has been made to give the reader some appreciation of the richness and history of differential equations through the use of historical notes. These notes, are intended to allow the reader to set the topic of differential equations in its proper perspective in the history of our culture. They can also be used by the instructor as an introduction to further discussions of mathematics.

DEPENDENCE OF CHAPTERS AND COURSE SUGGESTIONS

Since one cannot effectively cover all nine chatpers of this book during a one-semester or quarter course, the following dependence of chapters might be useful in organizing a course of study. Normally, one should think of this text as a one-semester book, although by covering all the material and working through a sufficient number of problems, it could be used for a two-semester course.

I often teach an introductory differential equations course for students of engineering and science. In that course I cover the first three chapters on first- and second-order equations, followed by Chapter 5 (the Laplace transform), Chapter 6 (systems), Chapter 8 (nonlinear equations), and part of Chapter 9 (partial differential equations). I generally spend a couple of days giving a rough overview of the omitted chapters: series solutions (Chapter 4) and difference equations (Chapter 7). For classes that contain mostly physics students who intended to take a follow-up course in partial differential equations, I cover Chapter 4 (series solutions) at the expense of some material on the Laplace transform.

I have on occasion used this book for a problems course in which I cover only Chapters 1, 2, and 3. Chapter 2 (first-order equations) contains a wide variety of problems that will keep any good student busy for an entire semester (some students have told me a lifetime).



ACKNOWLEDGMENTS

No textbook author can avoid thanking the authors of the many textbooks that have come before. A few of the textbooks to which I am indebted are: *Differential Equations* by Ralph Palmer Agnew, McGraw-Hill, New York, (1942), *Differential Equations* by Lester R. Ford, McGraw-Hill, New York, (1955), and *Differential Equations* by Lyman Kells, McGraw-Hill, New York, (1968).

I would also like to thank my advisor of more than 25 years ago, Ronald Guenther, who in addition to teaching me mathematics, taught me the value of rewriting.

I am grateful to the many people who contributed to this book at various stages of the project. The following people offered excellent advice, suggestions, and ideas as they reviewed the manuscript: Kevin T. Andrews, Oakland University; William B. Bickford, Arizona State University; Juan A. Gatica, University of Iowa; Peter A. Griffin, California State University, Sacramento; Terry L. Herdman, Virginia Polytechnic Institute and State University; Hidefumi Katsuura, San Jose State University; Monty J. Strauss, Texas Tech University; Peter J. Tonellato, Marquette University.

Finally, I am deeply grateful to the McGraw-Hill editors Jack Shira and Maggie Lanzillo, for their leadership and encouragement, and to Margery Luhrs and Richard Ausbum who have contributed to the project and worked so hard throughout the production process.

All errors are the responsibility of the author and I would appreciate having these brought to my attention. I would also appreciate any comments or suggestions from students and instructors.

Stanley J. Farlow

AN INTRODUCTION TO
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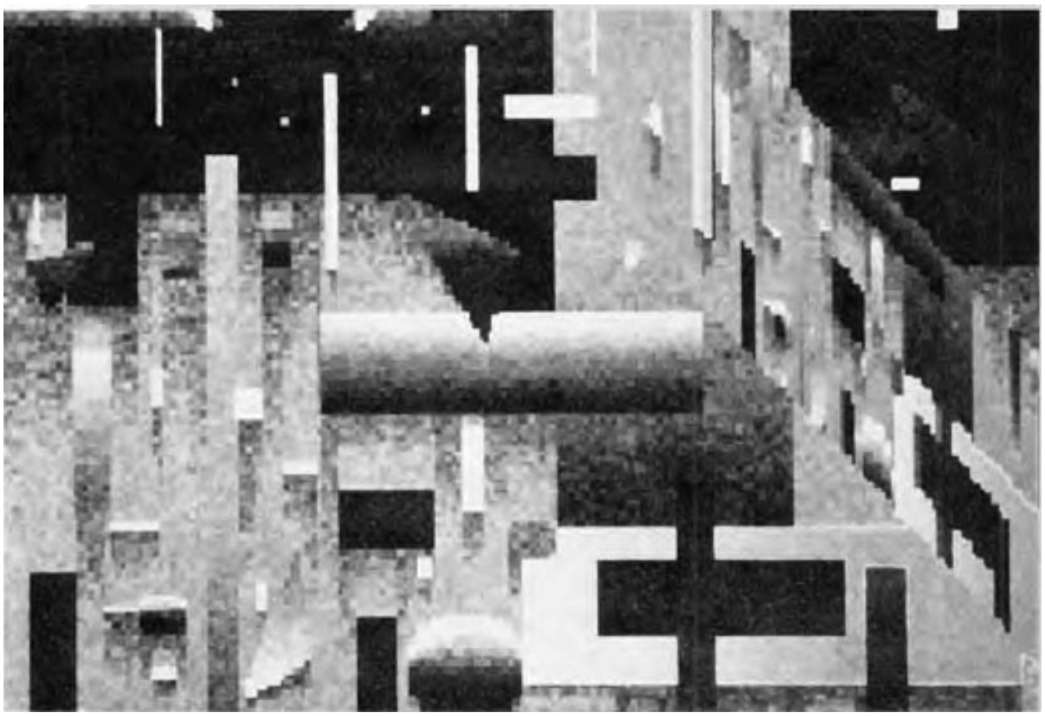
CHAPTER

1

Introduction to Differential Equations

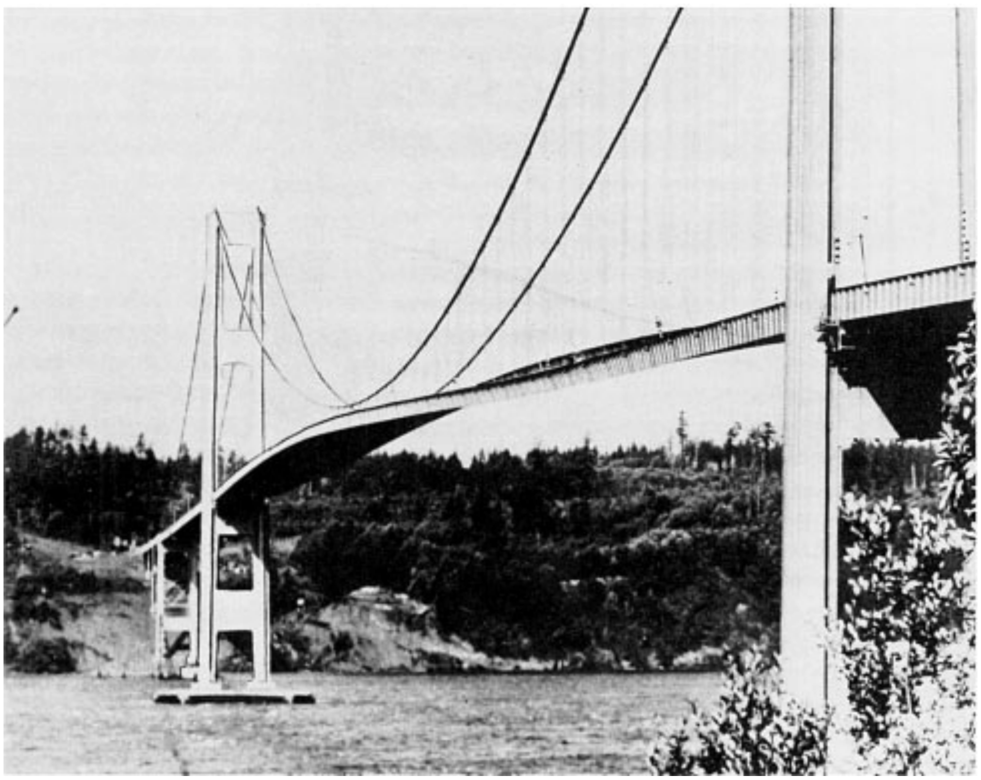


1.0 PROLOGUE

GALLOPING GERTIE

There was a lot of excitement in the air when on July 1, 1940, local dignitaries cut the ribbon that opened the Tacoma Narrows Bridge over

Puget Sound in the state of Washington, but the excitement didn't stop there. Because it tended to experience undulating vibrations in the slightest breeze, the bridge gained a great deal of attention and was nicknamed “Galloping Gertie.” Although one might have thought that people would have been afraid to cross the bridge, this was not so. People came from hundreds of miles just for the thrill of crossing “Gertie.” Although a few engineers expressed concern, authorities told the public that there was “absolutely nothing to worry about.” They were so sure of this that they even planned to drop the insurance on the bridge.



When Galloping Gertie collapsed into Puget Sound on November 7, 1940, bridge designers gained a new respect for nonlinear differential equations. (AP/Wide World Photos)

However, at about 7:00 A.M. on November 7, 1940, Gertie's undulations became more violent, and entire portions of the bridge began to heave wildly. At one time, one side of the roadway was almost 30 feet higher than the other. Then, at 10:30 A.M. the bridge began to crack up. Shortly thereafter it made a final lurching and twisting motion and then crashed into Puget Sound. The only casualty was a pet dog owned by a reporter who was crossing the bridge in his car. Although the reporter managed to reach safety by crawling on his hands and knees, clinging to the edge of the roadway, the dog lost its life.

Later, when local authorities tried to collect the insurance on the bridge, they discovered that the agent who had sold them the policy hadn't told the insurance company and had pocketed the $800,000 premium. The agent, referring to the fact that authorities had planned on canceling all policies within a week, wryly observed that if the “damn thing had held out just a little longer no one would have been the wiser.” The man was sent to prison for embezzlement. The collapse also caused embarrassment to a local bank, whose slogan was “As safe as the Tacoma Bridge.” After the bridge collapsed into Puget Sound, bank executives quickly sent out workers to remove the billboard.

Of course, after the collapse the government appointed all sorts of commissions of inquiry. The governor of the State of Washington made an emotional speech to the people of Washington proclaiming that “we are going to build the exact same bridge, exactly as before.” Upon hearing this, the famous engineer Theodor von Karman rushed off a telegram stating, “If you build the exact same bridge, exactly as before, it will fall into the same river, exactly as before.”

After the politicians finished their analysis of the bridge's failure, several teams of engineers from major universities began a technical analysis of the failure. It was the general consensus that the collapse was due to resonance caused by an aerodynamical phenomenon known as “stall flutter.”

Roughly, this phenomenon has to do with frequencies of wind currents agreeing with natural frequencies of vibration of the bridge. The phenomenon can be analyzed by comparing the driving frequencies of a differential equation with the natural frequencies of the equation.

FISHES, FOXES, AND THE NORWAY RAT

Although at one time Charlie Elton suspected that sunspot activity might be the cause of the periodic fluctuation in the rodent population in Norway, he later realized that this fluctuation probably had more to do with the ecological balance between the rats and their biological competitors.

The populations of many species of plants, fish, mammals, insects, bacteria, and so on, vary periodically due to boom and bust cycles in which they alternately die out and recover in their constant struggle for existance against their ecological adversaries. (Leonard Lee Rue Ill/Photo Researchers)

At about the same time, in the 1920s an Italian marine biologist, Umberto D’Ancona, observed that certain populations of fish in the northern Adriatic varied periodically over time. More specifically, he noted that when the population of certain *predator* fish (such as sharks, skates, and rays) was up, the population of their *prey* (herbivorous fish) was down, and vice versa. To better understand these “boom and bust” cycles, D’ Ancona turned to the famous Italian mathematician and differential equations expert Vito Volterra. What Volterra did was to repeat for biology what had been done in the physical sciences by Newton 300 years earlier. In general, he developed a mathematical theory for a certain area of biology; in particular, he developed a mathematical framework for the cohabitation of organisms. One might say that he developed the mathematical theory for the “struggle for existence” and that current research in ecological systems had its beginnings in the differential equations of Volterra.

WHERE WERE YOU WHEN THE LIGHTS WENT OUT?

Most readers of this book were probably pretty young during the New York City power failure of 1977 that plunged the entire northeastern section of the United States and a large portion of Canada into total darkness. Although the lessons learned from that disaster have led to more reliable power grids across the country, there is always the (remote) possibility that another failure will occur at some future time.

The problem is incredibly complicated. How to match the energy needs of the millions of customers with the energy output from the hundreds of generating stations? And this must be done so that the entire network remains synchronized at 60 cycles per second and the customer's voltage levels stay at acceptable levels! Everything would not be quite so difficult if demand remained constant and if there were never any breakdowns. As one system engineer stated, “It's easy to operate a power grid if nothing breaks down. The trick is to keep it working when you have failures.” However, there will always be the possibility of a generator breaking down or lightning hitting a transformer. And when this happens, there is always the possibility that the entire network may go down with it.

In any large scale system there is always the possibility that a failure in one part of the system can be propagated throughout the system. Systems of differential equations can be used to help understand the total dynamics of the system and prevent disasters. (Bill Gallery/Stock, Boston)

To help design large-scale power grids to be more reliable (stable), engineers have constructed mathematical models based on systems of differential equations that describe the dynamics of the system (voltages and currents through power lines). By simulating random failures the engineers are able to determine how to design reliable systems. They also use mathematical models to determine after the fact how a given failure can be prevented in the future. For example, after a 1985 blackout in Colombia, South America, mathematical models showed that the system would have remained stable if switching equipment had been installed to trip the transmission lines more quickly.

DIFFERENTIAL EQUATIONS IN WEATHER PATTERNS

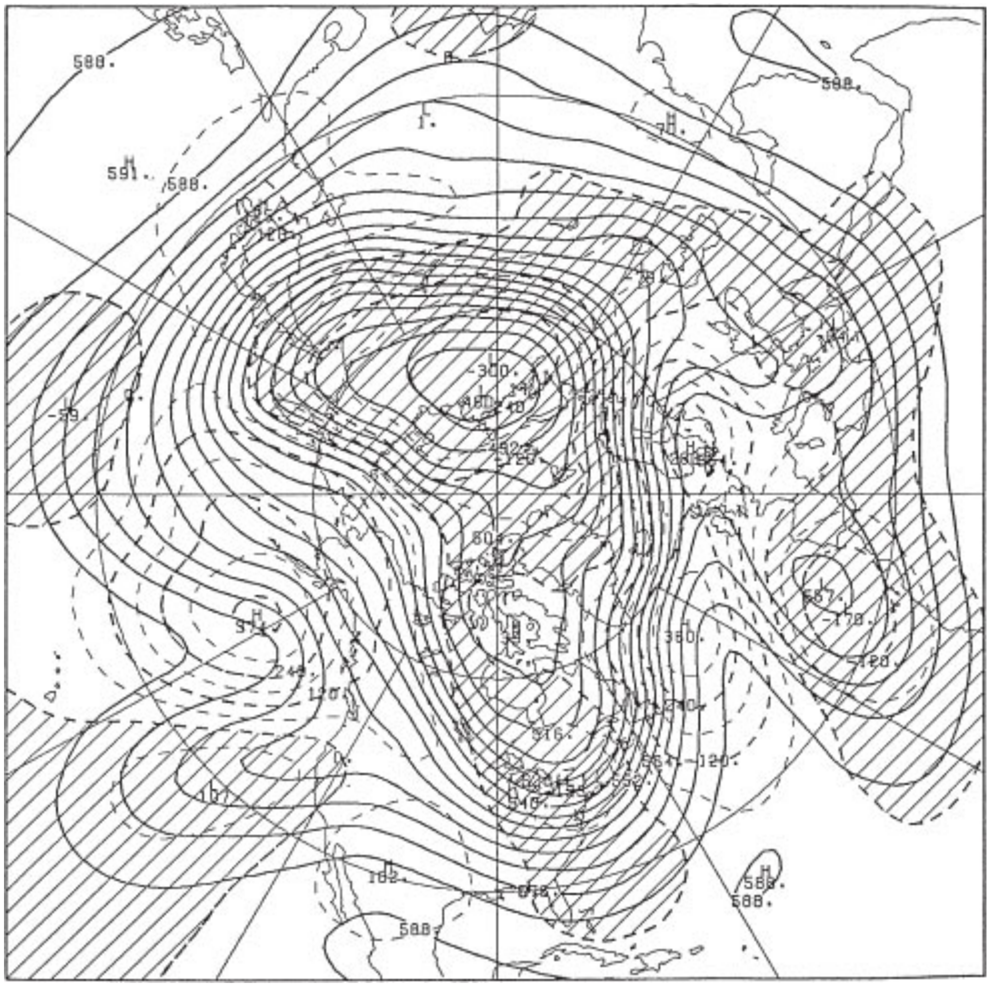
Meteorologist Edward Lorenz was not interested in the cloudy weather outside his M.I.T. office. He was more interested in the weather patterns he was generating on his new Royal McBee computer. It was the winter of 1961, and Lorenz had just constructed a mathematical model of convection patterns in the upper atmosphere based on a system of three nonlinear differential equations. In the early 1960s there was a lot of optimism in the scientific world about weather forecasting, and the general consensus was that it might even be possible in a few years to modify and control the weather. Not only was weather forecasting generating a great deal of excitement, but the techniques used in meteorology were also being used by physical and social scientists hoping to make predictions about everything from fluid flow to the flow of the economy.

Anyway, on that winter day in 1961 when Edward Lorenz came to his office, he decided to make a mathematical shortcut, and instead of running his program from the beginning, he simply typed into the computer the numbers computed from the previous day's run. He then turned on the computer and left the room to get a cup of coffee. When he returned an hour later, he saw something unexpected—something that would change the course of science.

The new run, which should have been the same as the previous day's run, was completely different. The weather patterns generated on this day were completely different from the patterns generated on the previous day, although their initial conditions were the same.

Initially, Lorenz thought he had made a mistake and keyed in the wrong numbers, or maybe his new computer had a malfunction. How else could he explain how two weather patterns had diverged if they had the same initial conditions? Then it came to him. He realized that the computer was using *six-place* accuracy, such as 0.209254, but only *three places* were displayed on the screen, such as 0.209. So when he typed in the new numbers, he had entered only *three decimal places*, assuming that one part in a thousand was not important. As it turned out insofar as the differential equations were concerned, it was *very* important.

The "chaotic" or "randomlike behavior" of those differential equations was so sensitive to their initial conditions that *no* amount of error was tolerable. Little did Lorenz know it at the time, but these were the differential equations that opened up the new subject of *chaos*. From this point on scientists realized that the prediction of such complicated physical phenomena as the weather was impossible using the classical methods of differential equations and that newer theories and ideas would be required. Paradoxically, chaos theory provides a way to see the *order* in a chaotic system.



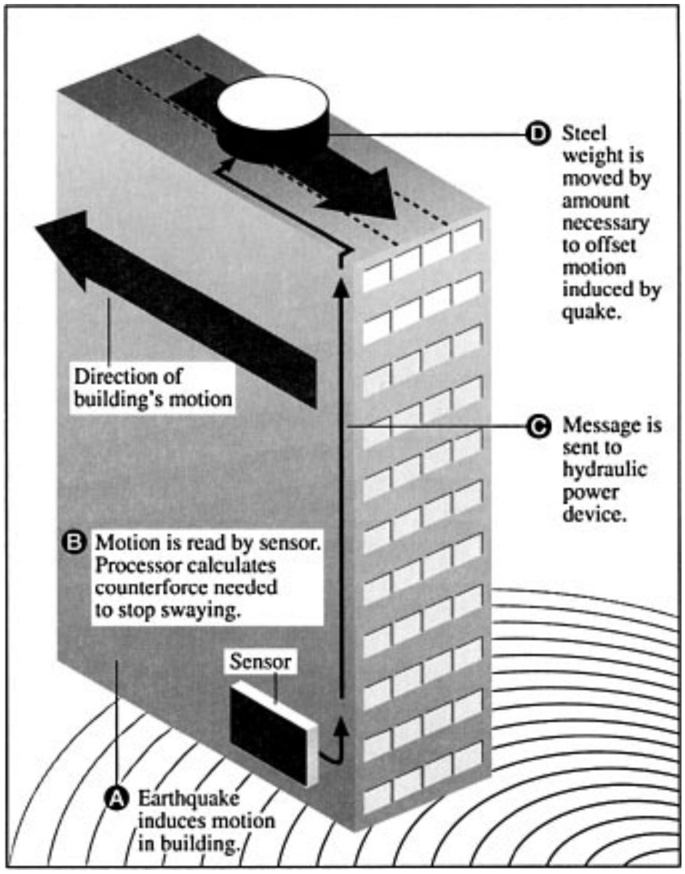
The future success of accurate long-run weather predictions is not completely clear. The accurate determination of long-term weather patterns could well depend on new research in dynamical systems and differential equations. (Courtesy National Meteorological Center)

ENGINEERS TEACH SMART BUILDING TO FOIL QUAKES*

Engineers and applied mathematicians are now designing self-stabilizing buildings that, instead of swaying in response to an earthquake, actively suppress their own vibrations with computer-controlled weights. (See Figure 1.1.) In one experimental building, the sway was said to be reduced by 80 percent.

During an earthquake, many buildings collapse when they oscillate naturally with the same frequency as seismic waves traveling through the earth, thus amplifying their effect, said Dr. Thomas Heaton, a seismologist at the U.S. Geological Survey in Pasadena, California. Active control systems might prevent that from happening, he added.

Figure 1.1 How a self-stabilizing building works. Instead of swaying in response to an earthquake, some new buildings are designed as machines that actively suppress their own vibrations by moving a weight that is about 1 percent of the building's weight.



One new idea for an active control system is being developed by the University of Southern California by Dr. Sami Masri and his colleagues in the civil engineering department. When wind or an earthquake imparts energy to the building, Dr. Masri said, it takes several seconds for the oscillation to build up to potentially damaging levels. Chaotic theory of differential equations, he said, suggests that a random source of energy should be injected into this rhythmic flow to disrupt the system.

At the present time, two new active stabilizing systems are to be added to existing buildings in the United States that sway excessively. Because the owners do not want their buildings identified, the names of the buildings are kept confidential.

Bridges and elevated highways are also vulnerable to earthquakes. During the 1989 San Francisco earthquake (the “World Series” earthquake) the double-decker Interstate 880 collapsed, killing several people, and the reader might remember the dramatic pictures of a car hanging precariously above San Francisco Bay where a section of the San Francisco-Oakland Bay Bridge had fallen away. Less reported was the fact that the Golden Gate Bridge might also have been close to going down. Witnesses who were on the bridge during the quake said that the roadbed underwent wavelike motions in which the stays connecting the roadbed to the overhead cables alternately loosened and tightened “like spaghetti.” The bridge oscillated for about a minute, about four times as long as the actual earthquake. Inasmuch as an earthquake of up to *ten times* this magnitude (the “big one”) is predicted for California sometime in the future, this experience reinforces our need for a deeper understanding of nonlinear oscillations in particular and nonlinear differential equations in general.

1.1 BASIC DEFINITIONS AND CONCEPTS

PURPOSE

To introduce some of the basic terminology and ideas that are necessary for the study of differential equations. We introduce the concepts of

#x2022; ordinary and partial differential equations,
- order of a differential equation,
- linear and nonlinear differential equations.

THE ROLE OF DIFFERENTIAL EQUATIONS IN SCIENCE

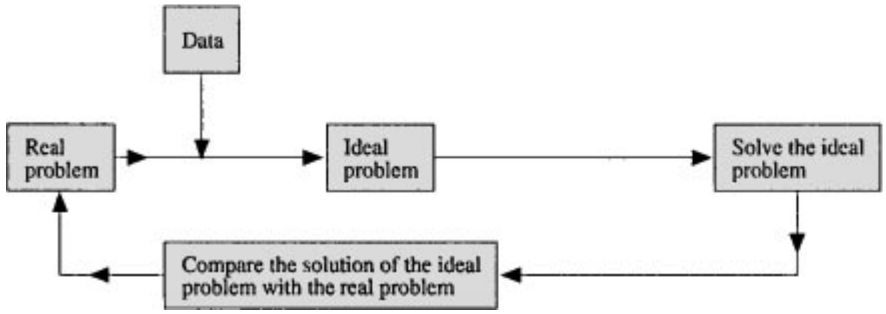
Before saying what a differential equation *is*, let us first say what a differential equation *does* and how it is used. Differential equations can be used to describe the amount of money in a savings bank, the orbit of a spaceship, the amount of deformation of elastic structures, the description of radio waves, the size of a biological population, the current or voltage in an electrical circuit, and on and on. *In fact, differential equations can be used to explain and predict new facts for about everything that changes continuously.* In more complex systems we don't use a single differential equation, but a *system* of differential equations, as in the case of an electrical network of several circuits or in a chemical reaction with several interacting chemicals.

The *process* by which scientists and engineers use differential equations to understand physical phenomena can be broken down into three steps. First, a scientist or engineer defines a real problem. A typical example might be the study of shock waves along fault lines caused by an earthquake. To understand such a phenomenon, the scientist or engineer first *collects data*, maybe soil conditions, fault data, and so on. This first step is called **data collection.**

The second step, called the **modeling process**, generally requires the most skill and experience on the part of the scientist. In this step the scientist or engineer sets up an *idealized problem*, often involving a differential equation, which describes the real phenomenon as precisely as possible while at the same time being stated in such a way that mathematical methods can be applied. This idealized problem is called a **mathematical model** for the real phenomenon. In this book, mathematical models refer mainly to differential equations with initial and boundary conditions. There is generally a dilemma in constructing a good mathematical model. On one hand, a mathematical model may describe accurately the phenomenon being studied, but the model may be so complex that a mathematical analysis is extremely difficult. On the other hand, the model may be easy to analyze mathematically but may not reflect accurately the phenomenon being studied. The goal is to obtain a model that is sufficiently accurate to explain all the facts under consideration and to enable us to predict new facts but at the same time is mathematically tractable.

The third and last step is to solve mathematically the ideal problem (i.e., the differential equation) and compare the solution with the measurements of the real phenomenon. If the mathematical solution agrees with the observations, then the scientist or engineer is entitled to claim with some confidence that the physical problem has been "solved mathematically," or that the theory has been verified. If the solution does not agree with the observations, either the observations are in error or the model is inaccurate and should be changed. This entire process of how mathematics (differential equations in this book) is used in science is described in Figure 1.2.

Figure 1.2 Schematic diagram of a mathematical analysis of physical phenomena



WHAT IS A DIFFERENTIAL EQUATION?

Quite simply, a **differential equation** is an equation that relates the derivatives of an unknown function, the function itself, the variables by which the function is defined, and constants. If the unknown function depends on a single real variable, the differential equation is called an **ordinary differential equation.** The following equations illustrate four well-known ordinary differential equations.

$$\frac{dy}{dx} + y = y^2 \qquad \text{(Bernoulli's equation)} \qquad (1a)$$

$$\frac{d^2y}{dx^2} = xy \qquad \text{(Airy's equation)} \qquad (1b)$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)y = 0 \qquad \text{(Bessel's equation)} \qquad (1c)$$

$$\frac{d^2y}{dx^2} - (1 - y^2)\frac{dy}{dx} + y = 0 \qquad \text{(Van der Pol's equation)} \qquad (1d)$$

In these differential equations the unknown quantity $y = y\{x)$ is called the **dependent variable**, and the real variable, x, is called the **independent variable.**

In this book, derivatives will be often represented by primes and higher derivatives sometimes by superscripts in parentheses. For example,

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''' = y^{(3)}, \quad \ldots \qquad (2)$$

Differential equations are as varied as the phenomena that they describe.* For this reason it is convenient to classify them according to certain mathematical properties. In so doing, we can better organize the subject into a coherent body of knowledge.

In addition to ordinary differential equations,† which contain ordinary derivatives with respect to a single independent variable, a **partial**

differential equation is one that contains partial derivatives with respect to more than one independent variable. For example, Eqs. (1a)–(1d) above are ordinary differential equations, whereas Eqs. (3a)–(3d) below are partial differential equations.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \text{(flux equation)} \tag{3a}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{(heat equation)} \tag{3b}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \text{(wave equation)} \tag{3c}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace's equation)} \tag{3d}$$

HOW DIFFERENTIAL EQUATIONS ORIGINATE

Inasmuch as derivatives represent rates of change, acceleration, and so on, it is not surprising to learn that differential equations describe many phenomena that involve motion. The most common models used in the study of planetary motion, the vibrations of a drumhead, or evolution of a chemical reaction are based on differential equations. In summary, differential equations originate whenever some universal law of nature is expressed in terms of a mathematical variable and at least one of its derivatives.

ORDER OF A DIFFERENTIAL EQUATION

Differential equations are also classified according to their order. The **order** of a differential equation is simply the *order* of the highest derivative that occurs in the equation. For example,

$$\frac{dy}{dx} - 3y = 2 \qquad \text{(first-order)} \tag{4a}$$

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 3y = 0 \qquad \text{(second-order)} \tag{4b}$$

$$y\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 0 \qquad \text{(first-order)} \tag{4c}$$

$$\frac{d^4y}{dx^4} - y = 0 \qquad \text{(fourth-order)} \tag{4d}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{(second-order)} \tag{4e}$$

DEFINITION: Ordinary Differential Equation

An nth-order ordinary differential equation is an equation that has the general form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{5}$$

where the primes denote differentiation with respect to x, that is, $y' = \partial y/\partial x$, $y'' = d^2y/dx^2$. and so on.

LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS

Some of the most important and useful differential equations that arise in applications are those that belong to the class of **linear differential equations**. Roughly, this means that they do not contain products, powers, or quotients of the unknown function and its derivatives. More precisely, it means the following.

DEFINITION: Linear Differential Equation

An nth-order ordinary differential equation is **linear** when it can be written in the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = f(x) \quad (a_0(x) \neq 0)$$

The functions $a_0(x)$, …, $a_n(x)$, are called the **coefficients** of the differential equation, and $f(x)$ is called the **nonhomogeneous term**. When the coefficients are constant functions, the differential equation is said to have **constant coefficients**. Unless it is otherwise stated, we shall always assume that the coefficients are continuous functions and that $a_0(x) \neq 0$ in any interval in which the equation is defined. Furthermore, the differential equation is said to be **homogeneous** if $f(x) \equiv 0$ and **nonhomogeneous** if $f(x)$ is *not* identically zero.

Finally, an ordinary differential equation that cannot be written in the above general form is called a **nonlinear ordinary differential equation.**

Some examples of linear and nonlinear differential equations are the following:

$$\frac{dy}{dx} = xy + 1 \qquad \text{(linear)} \tag{6a}$$

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y^2 = 0 \qquad \text{(nonlinear)} \tag{6b}$$

$$a_0(x)\frac{dy}{dx} + a_1(x)\,y = g(x) \qquad \text{(linear)} \tag{6c}$$

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = g(x) \qquad \text{(linear)} \tag{6d}$$

$$\frac{dy}{dx} = -\frac{x}{y} \qquad \text{(nonlinear)} \tag{6e}$$

$$yy'' + y' + y = 1 \qquad \text{(nonlinear)} \tag{6f}$$

Table 1.1 summarizes the above ideas. Note that in Table 1.1 the concepts of being homogeneous or having constant coefficients have no relevance for nonlinear differential equations.

Table 1.1 Classification of Differential Equations

Differential equation	Linear or nonlinear	Order	Homogeneous or nonhomogeneous	Constant or variable coefficients
$\frac{dy}{dx} + xy = 1$	Linear	1	Nonhomogeneous	Variable
$\frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = x$	Nonlinear	2	*	*
$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y^2 = 0$	Nonlinear	2	*	*
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$	Linear	2	Homogeneous	Constant
$\frac{d^2y}{dx^2} + y = \sin y$	Nonlinear	2	*	*
$\frac{d^4y}{dx^4} + 3y = \sin x$	Linear	4	Nonhomogeneous	Constant

PROBLEMS: Section 1.1

For Problems 1–10, classify each differential equation according to the following categories: order; linear or nonlinear; constant or variable coefficients; homogeneous or nonhomogeneous.

1. $\dfrac{dy}{dx} + xy^2 = 1$

2. $x\dfrac{dy}{dx} + y = \sin x$

3. $e^x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

4. $y\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 1$

5. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

6. $\dfrac{d^3y}{dx^3} + y = \sin x$

7. $\dfrac{d^2u}{dt^2} + t\dfrac{du}{dt} + 3u = 1$

8. $\dfrac{d^2w}{dt^2} - w^2\dfrac{dw}{dt} + w = 0$

9. $\dfrac{d^2v}{dt^2} = t^2v$

10. $\dfrac{d^2y}{dt^2} + y^2 = 0$

Problems 11–17 list differential equations that arise in the mathematical formulation of pure and applied science. Classify each equation according to order, number of independent variables, whether it is ordinary or partial, and whether it is linear or nonlinear. For linear equations, tell whether the equation has constant coefficients and whether it is homogeneous or nonhomogeneous.

11. $y' = ky$ (unrestricted growth or decay equation)

12. $y'' + \omega^2 y = 0$ (simple harmonic motion equation)

13. $EI\,y^{(4)} = w(x)$ (deflection of beam equation)

14. $u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = 0$ (Laplace's equation, u_r, u_{rr}, and $u_{\theta\theta}$ denote partial derivatives)

15. $u_t + u_{xxx} - 6uu_x + \dfrac{1}{2t}u = 0$ (*KdV* equation from fluid dynamics)

16. $y'' + y + \epsilon y^3 = 0$ (Hill's equation for vibrating systems)

17. $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ (Legendre's wave equation)

18. Gotcha Beginning students of differential equations are often confused as to whether a linear differential equation is homogeneous or nonhomogeneous. Problems (a)–(d) often confuse the beginner. Can you say for sure which of the linear equations in (a)–(d) are homogeneous?

(a) $y' - xy - 1 = 0$

(b) $xy' + y + \sin x = 0$

(c) $y'' + x + y = 0$

(d) $xy'' + xe^{x} + y' = 0$

Keeping a Scholarly Journal (Read This)

One cannot help but be impressed with the large number of important English naturalists who lived during the nineteenth century. Of course, there was Darwin, but there were also Wallis, Eddington, Thompson, Haldane, Fisher, Jevons, Fechner, Galton, and many more. If one studies the works of these eminent scholars, one cannot help but be impressed with the manner in which they paid attention to scientific details. Part of that attention to scientific details was the keeping of detailed journals in which they recorded their observations and impressions. These journals provided not only a means for storing data, but a means for exploring their thoughts and ideas. *They in fact learned through writing.*

Over the past 100 years, journal keeping has declined in popularity, but in recent years there has been a renaissance in the "learning through writing" movement. A few people are beginning to realize that writing is an important *learning* tool as well as a means of communication.

In this book we give the reader the opportunity to explore thoughts and ideas through writing. We only require that the reader *possess* a bound journal* in which daily entries are made. Each entry should be *dated* and, if useful, given a short title. There are no rules telling you what to include in your journal or how to write. The style of writing is strictly free form—don't worry about punctuation, spelling, or form. You will find that if you make a conscientious effort to make a daily entry, your writing style will take care of itself.

The best time to make your entry is immediately after class. Some professors allow their students the last five minutes of class time for journal entries. You might spend five minutes writing about the day's lecture. You might focus on a difficult concept. Ask yourself what you don't understand. Realize that you are writing for yourself. No one cares about your journal except you.

19. Your First Journal Entry Spend ten minutes exploring your goals for this course. Do you think differential equations will be useful to you? How does the material relate to your career goals as you see them? Maybe summarize in your own language the material covered in this first section. Be sure to date your entry. There will be a journal entry suggestion at the end of each problem set. Good luck.

1.2 SOME BASIC THEORY

PURPOSE

To introduce more concepts that are necessary to the study of differential equations. In particular, we will study

- explicit and implicit solutions,
- initial-value problems,
- the existence and uniqueness of solutions.

SOLUTIONS OF DIFFERENTIAL EQUATIONS

The general form of an *n*th-order ordinary differential equation can be written as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \tag{1}$$

where F is a function of the independent variable x, the dependent variable y, and the derivatives of y up to order n. We assume that x lies in an interval I that can be any of the usual types: (a, b), $[a, b]$, (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$, and so on. Often, it is possible to solve algebraically for the highest-order derivative in the differential equation and write it as

$$\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right) \tag{2}$$

where f is a function of $x, y, y', \ldots, y^{(n)}$

One of the main reasons for studying differential equations is to learn how to "solve" a differential equation.

HISTORICAL NOTE

The origins of differential equations go back to the beginning of the calculus to the work of Isaac Newton (1642–1727) and Gottfried Wilhelm von Leibniz (1646–1716). Newton classified first-order differential equations according to the forms $\partial y/\partial x = f(x)$, $\partial y/\partial x = f(y)$, or $\partial y/\partial x = f(x, y)$. He actually developed a method for solving $\partial y/\partial x = f(x, y)$ using infinite series when $f(x, y)$ is a polynomial in x and y. A simple example would be $\partial y/\partial x = 1 + xy$. Can you find an infinite series $y = y(x)$ that satisfies this equation?

This brings us to the concept of the solution of a differential equation.

DEFINITION: Solution of a Differential Equation*

A **solution** of an nth-order differential equation is an n times differentiable function $y = y(x)$, which, when substituted into the equation, satisfies it identically over some interval $a < x < b$. We would say that the function y is a solution of the differential equation **over the interval** $a < x < b$.

Example 1

Verifying a Solution Verify that the function

$$y(x) = \sin x - \cos x + 1 \tag{3}$$

is a solution of the equation

$$\frac{d^2y}{dx^2} + y = 1 \tag{4}$$

for all values of x.

Solution

Clearly, $y(x)$ is defined on $(-\infty, \infty)$, and $y'' = -\sin x + \cos x$. Substituting y'' and y into the differential equation (4) yields the identity

$$\underbrace{(-\sin x + \cos x)}_{y''} + \underbrace{(\sin x - \cos x + 1)}_{y} = 1$$

for all $-\infty < x < \infty$.

Example 2

Verifying a Solution Verify that the function

$$y(x) = 3e^{2x}$$

is a solution of the differential equation

$$\frac{dy}{dx} - 2y = 0 \tag{5}$$

for all x.

Solution

Clearly, the function $y(x)$ is defined for all real x, and substitution of $y(x) = 3e^{2x}$ and $y'(x) = 6e^{2x}$ into the differential equation yields the identity

$$\underset{\substack{\uparrow \\ y'}}{(6e^{2x})} - 2\underset{\substack{\uparrow \\ y}}{(3e^{2x})} = 0 \tag{6}$$

In fact, note that *any* function of the form $y(x) = Ce^{2x\prime}$. where C is a constant, is a solution of this differential equation.

Example 3

Verifying a Solution Verify that both functions

$$y_1(x) = e^{5x} \quad \text{and} \quad y_2(x) = e^{-3x} \tag{7}$$

are solutions of the second-order equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 0 \tag{8}$$

for all real x.

Solution

Substituting $y_1(x) = e^{5x}$ into the equation gives

$$y'' - 2y' - 15y = \underset{\substack{\uparrow \\ y''}}{(25e^{5x})} - 2\underset{\substack{\uparrow \\ y'}}{(5e^{5x})} - 15\underset{\substack{\uparrow \\ y}}{(e^{5x})} = 0$$

Substituting $y_2(x) = e^{-3x}$ into the equation, we get

$$y'' - 2y' - 15y = \underset{\substack{\uparrow \\ y''}}{(9e^{-3x})} - 2\underset{\substack{\uparrow \\ y'}}{(-3e^{-3x})} - 15\underset{\substack{\uparrow \\ y}}{(e^{-3x})} = 0$$

Hence both functions satisfy the equation for all x.

IMPLICIT SOLUTIONS

We have just studied solutions of the form $y = y(x)$ that determine y directly from a formula in x. Such solutions are called **explicit solutions**, since they give y directly, or *explicitly*, in terms of x. On some occasions, especially for nonlinear differential equations, we must settle for the less convenient form of solution, $G(x, y) = 0$, from which it is impossible to deduce an *explicit representation* for y in terms of x. Such solutions are called **implicit solutions.**

DEFINITION: Implicit Solution

A relation $G(x, y) = 0$ is said to be an implicit solution of a differential equation involving x, y, and derivatives of y with respect to x if $G(x, y) = 0$ *defines* one or more explicit solutions of the differential equation.*

Example 4

Implicit Solution Show that the relation

$$x + y + e^{xy} = 0 \tag{9}$$

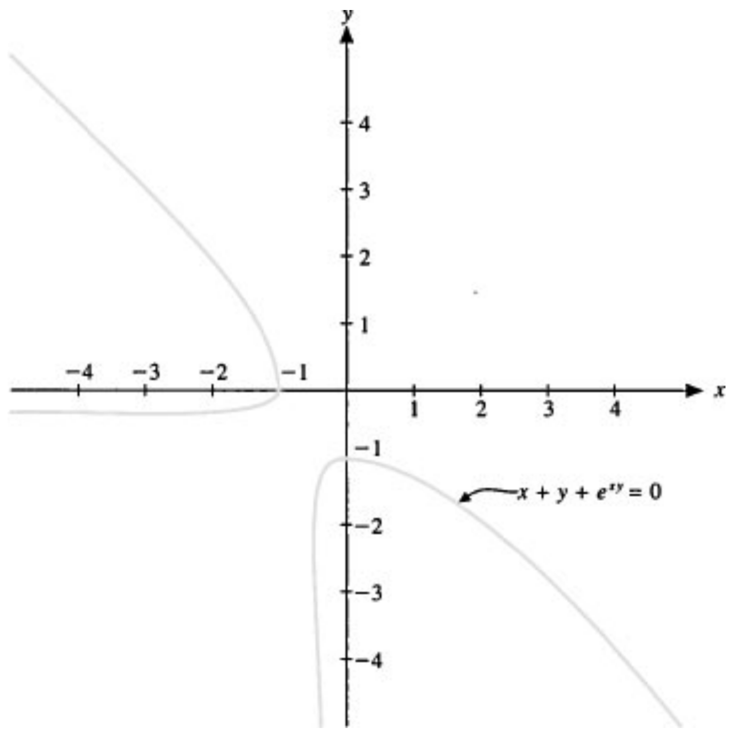
is an implicit solution of

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 \tag{10}$$

Solution

First, note that we cannot solve Eq. (9) for y in terms of x. However, a change in x in Eq. (10) results in a change in y, so we would expect that on some interval, Eq. (9) would define at least one function † $y = y(x)$. This is true in this case, and such a function $y = y(x)$ is also differentiable. See Figure 1.3.

Figure 1.3 Note that $x + y + e^{xy} = 0$ defines a function $y = y(x)$ on certain intervals and that this function is an explicit solution of the differential equation. Also, it can be shown that the slope $\partial y/\partial x$ of the tangent line at each point of the curve satisfies Eq. (9).



Once we know that the implicit relationship in Eq. (9) defines a differentiable function of x, we differentiate implicitly with respect to x. Doing this, we get

$$1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0 \tag{11}$$

which is equivalent to the differential equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 \tag{12}$$

Hence Eq. (9) is an implicit solution of Eq. (10).

Example 5
Implicit Solution Show that the relation

$$x^2 + y^2 - c = 0 \tag{13}$$

where c is a positive constant and is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{14}$$

on the open interval $(-c, c)$.

Solution

By differentiating Eq. (13) with respect to x we get

$$2x + 2y\frac{dy}{dx} = 0 \tag{15}$$

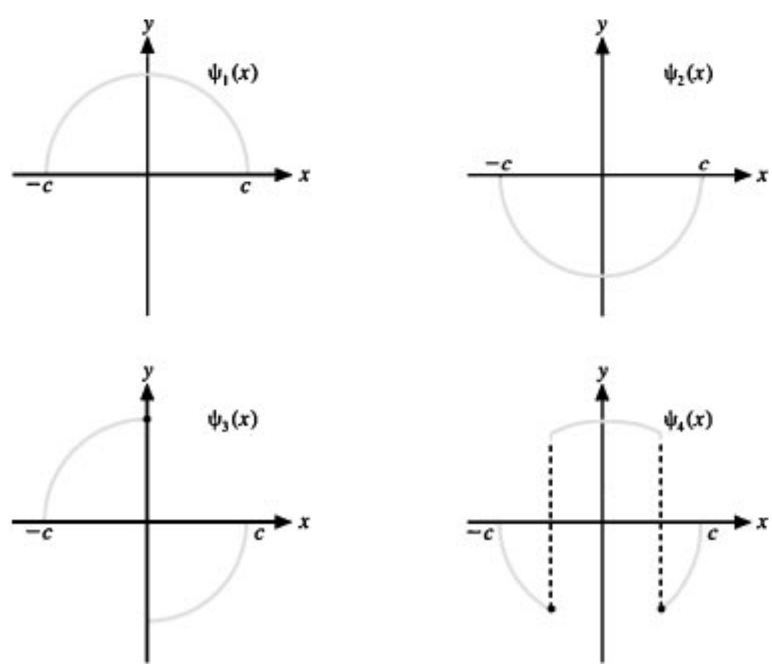
or $y' = -x/y$, which shows that Eq. (13) is an implicit solution of Eq. (14).

The geometric interpretation of this implicit solution is that the tangent line to the circle $x^2 + y^2 = c$ at the point (x, y) has slope $\partial y/\partial x = -x/y$. Also, note that there are many functions $y = y(x)$ that satisfy the implicit relation $x^2 + y^2 - c = 0$ on $(-c, c)$, and some of them are shown in Figure 1.4. However, the only ones that are continuous (and hence possibly differentiable) are

$$y(x) = +\sqrt{c - x^2} \qquad \text{and} \qquad y(x) = -\sqrt{c - x^2}$$

Note that $y'(-c)$ and $y'(c)$ do not exist, and so $y' = -x/y$ is an implicit relation only on the open interval $(-c, c)$. Hence from the implicit solution we are able to find two *explicit* solutions on the interval $(-c, c)$.

Figure 1.4 By taking portions of either the upper or lower semicircle of the circle $x^2 + y^2 = c^2$, one obtains a function $Y = y(x)$ that satisfies the relationship $x^2 + y^2(x) = c^2$ on the interval $(-c, c)$. A few of them are shown here.



COMMENT ON EXPLICIT VERSUS IMPLICIT SOLUTIONS

To better appreciate the use of the words “explicit” and “implicit,” we would say that $y = x + 1$ states *explicitly* that y is $x + 1$ and thus is called an explicit solution. On the other hand, the equation $y - x = 1$ does not state explicitly that y is $x + 1$ but only *implies* or states *implicitly* that y is $x + 1$. When a differential equation is solved, it is generally an explicit solution* that is desired.

THE INITIAL-VALUE PROBLEM

When solving differential equations in science and engineering, one generally seeks a solution that also satisfies one or more supplementary conditions such as initial or boundary conditions. The general idea is to first find *all* the solutions of the differential equation and then pick out the particular one that satisfies the supplementary condition(s).

DEFINITION: Initial-Value Problem

An **initial-value problem** for an *n*th-order equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

consists in finding the solution to the differential equation on an interval I that also satisfies the n **initial conditions**

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ y''(x_0) &= y_2 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

where x_0 ε I and …, y_{n-1} are given constants.

Note that in the special case of a first-order equation the only initial condition is $y(x_0) = y_0$, and in the case of a second-order equation the initial conditions are $y(x_0) = y_0$ and $y'(x_0) = y_1$.

The reason it is natural to specify n “side” conditions to accompany the nth-order linear differential equation lies in the fact that the general solution of the nth-order linear equation contains n arbitrary constants. Hence the n initial conditions will determine the constants, giving a unique solution to the initial-value problem.

Example 6

First-Order Instial-Value Problem Verify that $y(x) = e^{-x} + 1$ is a solution of the initial-value problem

$$y' + y = 1 \qquad y(0) = 2 \tag{16}$$

Solution

Computing $y'(x) = -e^{-x}$ and substituting $y(x)$ and $y'(x)$ into the differential equation, we get

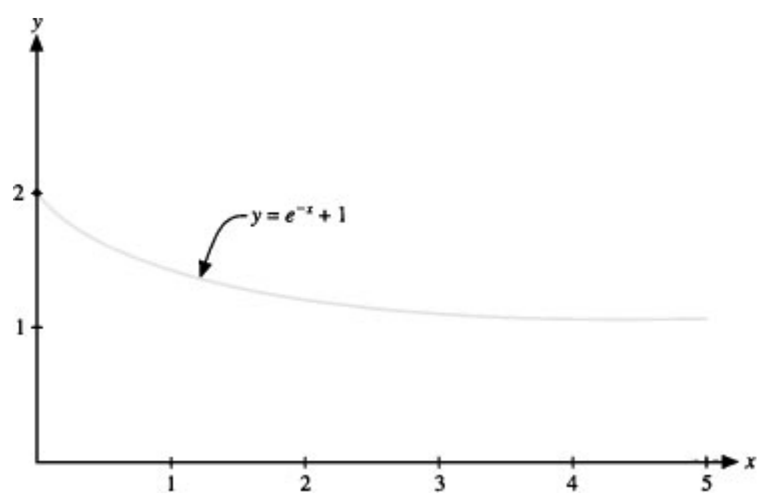
$$y' + y = -e^{-x} + (e^{-x} + 1) = 1$$

Hence $y(x)$ satisfies the differential equation. To verify that $y(x)$ also satisfies the initial condition, we observe that

$$y(0) = e^0 + 1 = 2$$

This solution is shown in Figure 1.5.

Figure 1.5 In Chapter 2 we will learn that $y(x) = ce^{-x} + 1$, where c is any constant, constitutes *all* the solutions of the equation $y' + y = 1$. However, the only one of these solutions that satisfies $y(0) = 2$ is $y(x) = e^{-x} + 1$.



Example 7

Initial-Value Problem Verify that $y(x) = \sin x + \cos x$ is a solution of the initial-value problem

$$y'' + y = 0 \qquad y(0) = 1 \qquad y'(0) = 1 \tag{17}$$

Solution

Computing $y'(x) = \cos x - \sin x$ and $y''(x) = -\sin x - \cos x$ and substituting these values into the differential equation, we get

$$y'' + y = (-\sin x - \cos x) + (\sin x + \cos x) = 0$$

Hence $y(x)$ satisfies the differential equation. To verify that $y(x)$ also satisfies the initial conditions, we observe that

$$\begin{aligned} y(0) &= \sin 0 + \cos 0 = 1 \\ y'(0) &= \cos 0 - \sin 0 = 1 \end{aligned}$$

This solution is drawn in Figure 1.6.

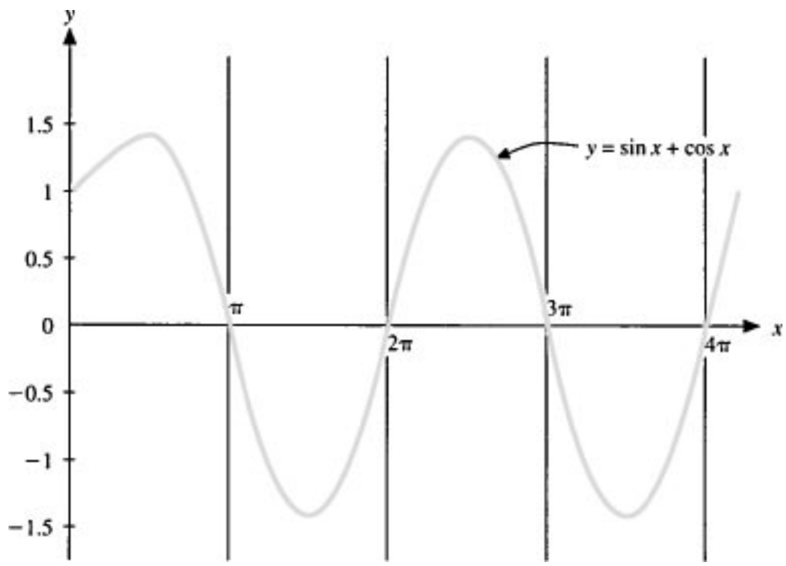
EXISTENCE AND UNIQUENESS OF SOLUTIONS

Although differential equations studied in applied work normally have solutions, it is clear that the equation

$$\left(\frac{dy}{dx}\right)^2 + 1 = 0$$

has none.* Inasmuch as some differential equations have solutions and some do not, it is important to know the conditions under which we know a solution exists. We state a fundamental *existence* and *uniqueness* result for the first-order initial-value problem.

Figure 1.6 Although it is clear that any function of the general form $y(x) = c_1, \sin x + c_2 \cos x$, where c_1 and c_2 are any constants, is a solution of $y'' + y = 0$, only $y(x) = \sin x + \cos x$ satisfies the conditions $y(0) = 1$ and $y'(0) = 1$.



Theorem 1.1 (PICARD'S THEOREM): Existence and Uniqueness for First-Order Equations

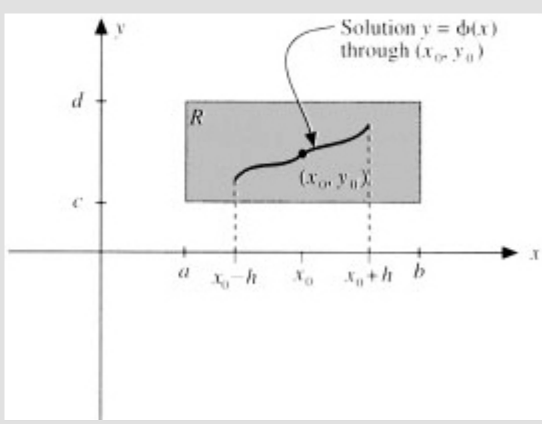
Assume that for the initial-value problem

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

the functions f and $\partial f/\partial y$ are continuous on some rectangle

$$R = \{(x, y): a < x < b, c < y < d\}$$

that contains the initial point (x_0, y_0). Under these conditions the initial-value problem has a unique solution $y = \Phi(x)$ on some interval $(x_0 - h, x_0 + h)$, where h is some positive number.



Note: Picard's theorem is one of the more popular existence and uniqueness theorems, since one only has to check the continuity of f and $\partial f/\partial y$, which is generally easy to do. A weakness of the theorem lies in the fact that it doesn't specify the size of the interval on which a solution exists without actually solving the differential equation.

Example 8

Picard's Theorem What does Picard's theorem guarantee about a solution to the initial-value problem

$$y' = y + e^{2x} \quad y(0) = 1 \tag{18}$$

Solution

Since $f(x, y) = y + e^{2x}$ and $\partial f/\partial y = 1$ are continuous in any rectangle containing the point (0, 1), the hypothesis of Picard's theorem is satisfied. Hence Picard's theorem guarantees that a unique solution of the initial-value problem exists in some interval $(-h, h)$, where h is a positive constant. We will learn how to solve this equation in Chapter 2 and see that the solution is $y(x) = e^{2x}$ for all $-\infty < x < \infty$. Picard's theorem tells us that this is the only solution of this initial-value problem.

HISTORICAL NOTE

Theorem 1.1 and other closely related existence theorems are generally associated with the name of Charles Emile Picard (1856–1941), one of the greatest French mathematicians of the nineteenth century. He is best known for his contributions to complex variables and differential equations. It is interesting to note that in 1899, Picard lectured at Clark University in Worcester, Massachusetts.

Example 9

Picard's Theorem What does Picard's theorem imply about a unique solution of the initial-value problem

$$y' = y^{1/3} \quad y(0) = 0$$

Solution

Here we have $f(x, y) = y^{1/3}$ and $\partial f/\partial y = \frac{1}{3}y^{2/3}$. Although $f(x, y)$ is continuous in the entire xy-plane, the partial derivative $\partial f/\partial y$ is not continuous on any rectangle that intersects the line $y = 0$. Hence the hypothesis of Picard's theorem fails, and so we cannot be *guaranteed** that there exists a unique solution to any problem whose initial value of y is zero. In Problem 45 at the end of this section we will see that there are in fact solutions to this problem—in fact, an infinite number of solutions.

GENERAL SOLUTION OF A DIFFERENTIAL EQUATION

The nature of the solutions of a differential equation is somewhat reminiscent of finding A DIFFERENTIAL antiderivatives in calculus. Remember that when finding an antiderivative, one obtains a function containing an arbitrary constant.* In solving a first-order differential equation $F(x, y, y') = 0$, the standard strategy is to obtain a *family* of curves $G(x, y, c) = 0$ containing one arbitrary constant c (called a parameter) such that each member of the family satisfies the differential equation. In the general case when solving an nth-order equation $F(x, y, y', \ldots, y^{(n)}) = 0$, we generally obtain an **n-parameter family of solutions** $G(x, y, c_1, c_2 \ldots, c_n) = 0$. A solution of a differential equation that is free of arbitrary parameters is called a specific or **particular solution.**

The theory of differential equations would be simplified if one could say that each differential equation of order n has an n-parameter family of solutions and that those are the only solutions. Normally, this is true as we will see for linear differential equations. However, there are some nasty nonlinear equations of order n that have an n-parameter family of solutions but still have a few more **singular solutions** hanging around the fringes, so to speak. For example, the nonlinear equation $(y')^2 + xy' = y$ has a one-parameter family of solutions $y(x) = cx + c^2$, but it still has one more *singular solution*, $y(x) = -x^2/4$. (See Problem 46 at the end of this section.) It is called a singular solution, since it cannot be obtained by assigning a specific value of c to $y(x) = cx + c^2$.

The fact that for some nonlinear differential equations, not all solutions of an nth-order equation are members of an n-parameter family has given rise to two schools of thought concerning the definition of a "general solution" of a differential equation. Some people say that the general solution of an nth-order differential equation is a family of solutions consisting of n essential† parameters. In this book we use a slightly broader definition of the general solution. We define the general solution of a differential equation to be the collection of *all* solutions of a differential equation. Period. If the only solutions of a differential equation consist of an n-parameter family of solutions, then the two definitions are the same. For those nasty nonlinear equations that have an n-parameter family of solutions *plus* a few more singular solutions, we will call the general solution the collection of both these types of solutions.

PROBLEMS: Section 1.2

For Problems 1–11, show that each function is a solution of the given differential equation. Assume that a and c are constants. For what values of the independent variable(s) is your solution defined?

Differential equation	Function
1. $\frac{dy}{dx} = ay$	$y = e^{ax}$
2. $\frac{dy}{dx} = y + e^x$	$y = xe^x$
3. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + a^2}} \quad (a \neq 0)$	$y = \sqrt{x^2 + a^2}$
4. $\frac{d^2y}{dx^2} + a^2 y = 0$	$y = c \sin ax$
5. $\frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2 - x\frac{dy}{dx} + y = 1 - x^2$	$y = x^2$
6. $\frac{1}{4}\left(\frac{d^2y}{dx^2}\right) - x\frac{dy}{dx} + y = -2$	$y = 2(x - 1)$
7. $(1 + x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 2$	$y = x^2$
8. $x(x - 2)\frac{d^2y}{dx^2} - 2(x - 1)\frac{dy}{dx} + 2y = 12x^5 - 30x^4$	$y = x^5$
9. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$
10. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$
11. $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$	$u(x, t) = f(x - at) + g(x + at)$ (where f and g are differentiable functions)

For Problems 12–16, show that the following relation defines an implicit solution of the given differential equation.

Differential equation	Relation
12. $yy' = e^{2x}$	$y^2 = e^{2x}$
13. $y' = \dfrac{xy}{x^2 + y^2}$	$2y^2 \ln y - x^2 = 0$
14. $2xyy' = x^2 + y^2$	$y^2 = x^2 - cx$
15. $y' = \dfrac{y^2}{xy - x^2}$	$y = ce^{y/x}$
16. $y' = \dfrac{xy}{x^2 - 1}$	$x^2 + cy^2 = 1$

Test Your Intuition

For Problems 17–25, see whether you can make an educated guess to find a solution of the given equation. After you have selected your candidate, check to see whether it satisfies the equation. Have you found all the solutions of the equation?

17. $\dfrac{dy}{dx} = y$ (When is the derivative equal to the function itself?)

18. $\dfrac{dy}{dx} = y^2$ (When is the derivative equal to the function squared?)

19. $\dfrac{dy}{dx} + y = 1$ (There is a solution staring you in the face.)

20. $\dfrac{dy}{dx} + y = e^x$ (Almost staring you in the face)

21. $\dfrac{dy}{dx} + y = ae^x$ (More interesting)

22. $\dfrac{dy}{dx} + ay = e^x$ (Kind of like Problem 21)

23. $\dfrac{dy}{dx} + \dfrac{1}{x}y = 0$ (A little tougher)

24. $\dfrac{d^2y}{dx^2} + y = 0$ Do you have all of them?)

25. $\dfrac{d^2y}{dx^2} - y = 0$ (Compare with Problem 24.)

26. **Simplest Differential Equations** The simplest of all differential equations are equations of the form

$$\frac{dy}{dx} = f(x), \quad \frac{d^2y}{dx^2} = f(x), \quad \ldots, \quad \frac{d^ny}{dx^n} = f(x), \quad \ldots \qquad (19)$$

that are studied in calculus. Recall that the solution of the first-order equation $y' = f(x)$ is simply the collection of antiderivatives of $f(x)$, or

$$y(x) = \int f(x)\, dx + c \qquad (20)$$

For Problems (a)–(e), solve the given first- or second-order differential equation for all $-\infty < x < \infty$ Where initial conditions are specified, solve the initial-value problem.

(a) $y' = 3$
(b) $y' = x^2$
(c) $y' = \sin x \quad y(0) = 0$
(d) $y'' = 1 \quad y(0) = 1 \quad y'(0) = 0$
(e) $y'' = \sin x \quad y(0) = 0 \quad y'(0) = 1$

Initial-Value Problems

For Problems 27–33, verify that the specified function is a solution of the given initial-value problem.

Differential equation	Initial condition(s)	Function
27. $y' + y = 0$	$y(0) = 2$	$y(x) = 2e^{-x}$
28. $y' = y^2$	$y(0) = 0$	$y(x) = 0$
29. $y' = y^2$	$y(1) = -1$	$y(x) = -1/x,\ x \in (0, \infty)$
30. $y' = -x/y$	$y(0) = 2$	$y(x) = \sqrt{4 - x^2},\ x \in (-2, 2)$
31. $y'' + 4y = 0$	$y(0) = 1 \quad y'(0) = 0$	$y(x) = \cos 2x$
32. $y'' - y = 0$	$y(0) = 1 \quad y'(0) = 0$	$y(x) = \cosh x$
33. $y'' + 3y' + 2y = 0$	$y(0) = 0 \quad y'(0) = 1$	$y(x) = e^{-x} - e^{-2x}$

34. No Solutions Why don't the following differential equations have real-valued solutions on *any* interval?

(a) $\left|\frac{dy}{dx}\right| + |x| + |y| + 1 = 0$
(b) $\left(\frac{dy}{dx}\right)^2 + 1 = -e^x$
(c) $|y'| + y^2 = -1$
(d) $\sin y' = 2$

35. Implicit Function Theorem The implicit function theorem states that if $G(x, y)$ has continuous first partial derivatives in a rectangle $R = \{(x, y): a < x < b,\ c < y < d\}$ containing a point (x_0, y_0), and if $G(x_0, y_0) = 0$ and $\partial G(x_0, y_0)/\partial y$ is not zero, then there exists a differentiable function $y = \Phi(x)$, defined on an interval $I = (x_0 - h, x_0 + h)$, that satisfies $G(x, \Phi(x)) = 0$ for all $x \varepsilon I$. The implicit function theorem provides the conditions under which $G(x, y) = 0$ defines y implicitly as a function of x. Use the implicit function theorem to verify that the relationship $x + y + e^{xy} = 0$ defines y implicitly as a function of x near the point $(0, -1)$.

Existence of Solutions

For Problems 36–43, determine whether Picard's theorem implies that the given initial-value problem has a unique solution on some interval containing the initial value of x.

36. $y' = x^2$ $\quad y(0) = 0$

37. $y' - y = 1$ $\quad y(0) = 3$

38. $y' = x^2 + y^2$ $\quad y(0) = 1$

39. $y' = x^3 - y^3$ $\quad y(0) = 0$

40. $y' = y^2$ $\quad y(1) = 1$

41. $yy' = x^2 + y^2$ $\quad y(0) = 1$

42. $y' = -x/y$ $\quad y(1) = 0$

43. $y' = y/x$ $\quad y(0) = 1$

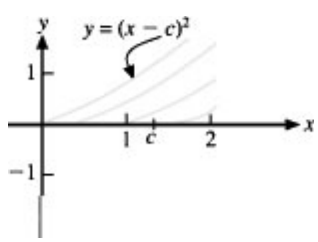
44. **A Strange Differential Equation** The initial-value problem

$$y' = 2\sqrt{y} \qquad y(0) = 0 \tag{21}$$

has an infinite number of solutions on the interval $[0, \infty)$.

(a)Show that $y(x) = x^2$ is a solution.

Figure 1.7 An infinite number of solutions of the initial-value problem $y' = 2\sqrt{y}$, $y(0) = 0$. This problem has long been a popular example for illustrating when Picard's theorem fails.



(b)Show that *any* function of the form

$$y(x) = \begin{cases} 0 & 0 \le x < c \\ (x - c)^2 & c \le x \end{cases}$$

where $c \ge 0$ is a solution of Eq. (21). See Figure 1.7.

(c)How do these results relate to Picard's existence and uniqueness theorem?

45. Hubbard's Empty Bucket* If you are given an empty bucket with a hole in it, can you determine when the bucket was full of water? Of course, the answer is no, but did you know that the reason is that a certain differential equation does not have a unique solution? It's true—The differential equation that describes the height h of the water satisfies **Torricelli's law**,

$$\frac{dh}{dt} = -k\sqrt{h} \tag{22}$$

where $k > 0$ is a constant depending on the cross section of the hole and the bucket.

(a)Show that Eq. (22) does not satisfy Picard's theorem.

(b)Find an infinite number of solutions of Eq. (22).

(c)Sketch the graphs of several solutions of Eq. (22) and discuss why you can't determine when the bucket was full if it is currently empty.

46. Singular Solution Given the first-order nonlinear equation

$$(y')^2 + xy' = y \tag{23}$$

verify the following.

(a)Each member of the one-parameter family of functions $y(x) = cx + c^2$, where c is a real constant, is a solution of Eq. (23).

(b)The function $y(x) = -1/4\ x^2$ cannot be obtained from $y = cx + c^2$ by any choice of c, yet it satisfies Eq. (23). It is a singular solution of Eq. (23).

47. Only One Parameter Show that it is possible to rewrite

$$y = c_1 + \ln(c_2 x)$$

in terms of only one parameter.

48. Delay Differential Equation A delay differential equation differs from the usual differential equation by the presence of a shift $x - x_0$ in the argument of the unknown function $y(x)$. These equations are much more difficult to solve than the usual differential equation, although much is known about them. Show that the simple delay differential equation

$$\frac{dy}{dx} = ay(x - b) \tag{24}$$

where $a \neq 0$ and b are given constants, has a solution of the form $y = Ce^{kx}$ for any constant C, provided that the constant k satisfies the transcendental equation $k = ae^{-bk}$. If you have access to a computer with a program to approximate solutions of transcendental equations, find an approximate solution to the equation $y' = y(x - 1)$ and sketch the graph of this solution.

49. **Journal Entry—How's Your Intuition?** Using your intuition, spend a few minutes and try to answer one of the following questions. How many solutions should there be to the equation $y' = ky$? What are all the solutions of the differential equation $y'' + y = 0$? Are there more solutions to second-order differential equations than to first-order equations? Can you find a solution of a differential equation by just finding the antiderivative of each term in the equation?

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* Based on an article taken from the *New York Times*, July 2, 1991.

* A differential equation is often named after the person who first studied it. For example, Van der Pol's equation listed here was first investigated by the Dutch radio engineer Balthasar Van der Pol (1889–1959), in studying oscillatory currents in electric circuits.

† Although differential equations should probably be called *derivative equations*, inasmuch as they contain derivatives, the term "differential equations" *(aequatio differential*) was coined by Gotfried Leibniz in 1676 and is used universally today.

* A good leather-bound journal can be purchased in any office supply store for about $10. Regular notebook paper can be inserted into these journals.

* Solutions of differential equations are sometimes called *integrals* of the differential equations, since they are more or less an extension of the process of integration in calculus.

* Realize that even though you may not be able to actually solve the equation $G(x, y) = 0$ for y, thus obtaining a formula in x, nevertheless, any change in x still results in a corresponding change in y. Thus the expression $G(x, y) = 0$ gives rise to at least one function $y = y(x)$, even though you cannot find a formula for it. The exact conditions under which $G(x, y) = 0$ gives rise to a function $y = y(x)$ are known as the **implicit function theorem.** Details of this theorem can be found in most textbooks of advanced calculus.

† The implicit function theorem provides exact conditions under which an implicit relation $G(x, y) = 0$ defines y as a function of x. See Problem 35 at the end of this section.

* The importance of implicit solutions lies in the fact that some methods of solution do not lead to explicit solutions, but only to implicit solutions. Thus implicit solutions might be thought of as "better than nothing." In some areas of nonlinear differential equations, implicit solutions are the *only* solutions obtainable and in this context are referred to simply as solutions.

* Although the equation has no real-valued solution, it does have a complex-valued solution. A study of complex-valued solutions of differential equations is beyond the scope of this book.

* Remember, the conditions stated in Picard's theorem are *sufficient* but not *necessary.* If the conditions stated in Picard's theorem hold, then there will exist a unique solution. However, if the conditions stated in the hypothesis of Picard's theorem do *not* hold, then nothing is known: The initial-value problem may have either (a) no solution, (b) more than one solution, or (c) a unique solution.

* For example, the antiderivative of $f(x) = 2x$ is $F(x) = x^2 + c$, where c is an arbitrary constant.

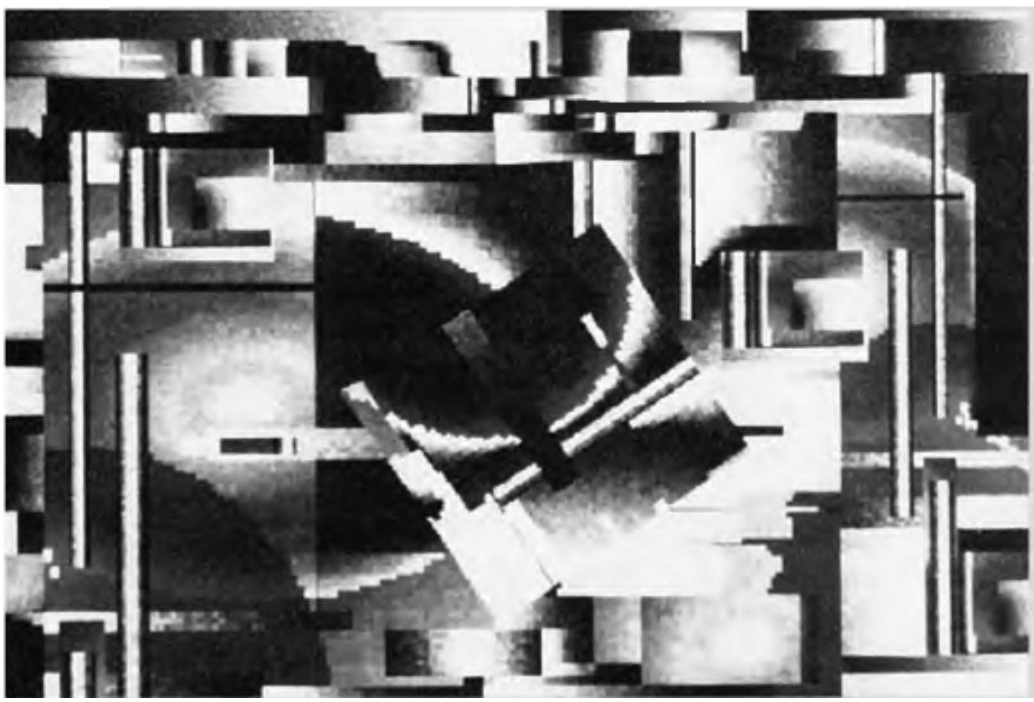
† By *essential* parameters or constants we mean just that. For instance, we could write the solution of the equation $y' = 1$ as either $y = x + c$ or $y = x + c_1 + c_2$. The two constants in second form are "fraudulent," since we could let $c = c_1 + c_2$. It is more difficult to determine the number of essential constants in other equations, such as $y = c_1 + \text{In } c_2 x$. How many do you think there are? Be careful, there is really only one! See Problem 47.

* This interesting problem is based on an example taken from *Differential Equations: A Dynamical Systems Approach* by J. H. Hubbard and B. H. West (Springer-Verlag, New York, 1989).

CHAPTER

2

First-Order Differential Equations



2.1 FIRST-ORDER LINEAR EQUATIONS

PURPOSE

To solve the general first-order linear differential equation

$$y' + p(x)y = f(x)$$

using the integrating factor method. We will construct a function $\mu(x)$ that satisfies $\mu(y' + py) = (\mu y)'$, thus allowing the equation to be integrated.

The general first-order linear differential equation can be written as

$$a_0(x)\frac{dy}{dx} + a_1(x)y = F(x) \tag{1}$$

where $a_0(x)$, $a_1(x)$, and $F(x)$ are given functions of x defined on some given interval* I. Inasmuch as we always assume that $a_0(x) \neq 0$ for all $x \in I$, it is convenient to divide by $a_0(x)$ and rewrite the equation as

$$y' + p(x)y = f(x) \tag{2}$$

where $p(x) = a_1(x)/a_0(x)$ and $f(x) = F(x)/a_0(x)$. If we assume that the functions $p(x)$ and $f(x)$ are continuous for x belonging to the interval I, then Picard's theorem guarantees the existence of a unique solution to Eq. (2) in some subinterval satisfying arbitrary initial conditions $y(x_0) = y_0$, where x_0 belongs to I. (See Problem 37 at the end of this section.) The goal of this section is to find the general solution (all solutions) of Eq. (2). In the case of first-order equations the general solution will contain one arbitrary constant.

INTEGRATING FACTOR METHOD (Constant Coefficients)

Before solving Eq. (2), however, we will solve the simpler equation

$$y' + ay = f(x) \tag{3}$$

where we have replaced $p(x)$ by the constant a. The idea behind the integrating factor method is the simple observation that

$$e^{ax}(y' + ay) = \frac{d}{dx}(e^{ax}y) \tag{4}$$

which turns the differential equation (3) into a “calculus problem.” To see how this method works, multiple each side of Eq. (3) by e^{ax}, getting

$$e^{ax}(y' + ay) = f(x)e^{ax}$$

which, using the fundamental property (4), reduces to

$$\frac{d}{dx}(e^{ax}y) = f(x)e^{ax} \tag{5}$$

This equation can now be integrated directly, and we get

$$e^{ax}y = \int f(x)e^{ax}dx + c$$

where c is an arbitrary constant and the integral sign refers to any antiderivative of $f(x)e^{ax}$. Solving for y gives

$$\boxed{\boldsymbol{y(x) = e^{-ax}\int f(x)e^{ax}\,dx + ce^{-ax}}} \tag{6}$$

We now solve the general first-order equation

$$y' + p(x)y = f(x) \tag{7}$$

where $p(x)$ is assumed to be a continuous function. The general idea is motivated by the constant coefficient equation; we seek a function $\mu(x)$. called an **integrating factor**, that satisfies

$$\mu(x)\,[y' + p(x)y] = \frac{d}{dx}\{\mu(x)y(x)\} \tag{8}$$

To find $\mu(x)$, we carry out the differentiation on the right-hand side and simplify, getting

$$\mu(x)y' + \mu(x)p(x)y = \mu'(x)y + \mu(x)y'$$

If we now assume that $y(x) \neq 0$, we arrive at

$$\mu'(x) = p(x)\mu(x)$$

But we can find a solution $\mu(x) > 0$ by separating variables, getting

$$\begin{aligned} \frac{\mu'(x)}{\mu(x)} &= p(x) \\ \ln \mu(x) &= \int p(x)\,dx \\ \mu(x) &= e^{\int p(x)\,dx} \end{aligned} \tag{9}$$

Note: Since $\int p(x)\,dx$ denotes the *collection* of *all* antiderivatives of $p(x)$, it contains an arbitrary additive constant. Hence $\mu(x)$ contains an arbitrary *multiplicative* constant. However, since we are interested in finding only one integrating factor, we will pick the multiplicative constant to be 1.

Now that we *know* the integrating factor, we simply multiply each side of Eq. (7) by the integrating factor (9), getting

$$\mu(x)[y' + p(x)y] = \mu(x)\,f(x)$$

But from the property $\mu(x)\,[y' + p(x)\,y] = [\mu(x)y]'$ we have

$$[\mu(x)\,y]' = \mu(x)\,f(x) \tag{10}$$

We can now integrate Eq. (10), getting

$$\mu(x)\,y(x) = \int \mu(x)\,f(x)\,dx + c \tag{11}$$

and since $\mu(x) \neq 0$, we can solve for $y(x)$ algebraically, getting

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\,f(x)\,dx + \frac{c}{\mu(x)} \tag{12}$$

We summarize these ideas, which give rise to the **integrating factor method** for solving the general first-order linear equation.

Integrating Factor Method

To solve the first-order linear differential equation

$$y' + p(x)y = f(x)$$

on a given interval I, perform the following steps.

Step 1 (Find the Integrating Factor). Find the integrating factor

$$\mu(x) = e^{\int p(x)\,dx}$$

where $\int p(x)\,dx$ represents *any* antiderivative of $p(x)$. Normally, pick the arbitrary constant in the antiderivative to be zero. Note that $\mu(x) \neq 0$ for $x \in I$.

Step 2 (Multiply by the Integrating Factor). Multiply each side of the differential equation by the integrating factor to get

$$e^{\int p(x)\,dx}\,(y' + p(x)\,y) = f(x)\,e^{\int p(x)\,dx}$$

which will always reduce to

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}\,y(x)\right) = f(x)\,e^{\int p(x)\,dx}$$

Step 3 (Find the Antiderivative). Integrate the equation from Step 2 to get

$$e^{\int p(x)\,dx}\, y(x) = \int f(x)\, e^{\int p(x)\,dx}\, dx + c$$

Step 4 (Solve for y). Solve the equation found in Step 3 for y(x) to get the general solution

$$y = e^{-\int p(x)\,dx} \int f(x)\, e^{\int p(x)\,dx}\, dx + ce^{-\int p(x)\,dx} \qquad (13)$$

Notes:

1. We have shown that if $y' + p(x)\, y = f(x)$ has a solution, it must be of the form in Eq. (13). Conversely, it is a straightforward matter to verify that Eq. (13) also constitutes a one-parameter family of solutions of the differential equation.
2. One could memorize the formula for the general solution of the general first-order linear equation. However, it is easier to simply remember that multiplication of the differential equation by $\mu(x)$ turns the differential equation into an "ordinary" antiderivative problem of the type one studies in calculus. The following examples illustrate the integrating factor method.
3. For first-order *linear* differential equations a one-parameter family of solutions constitutes *all* the solutions of the equation and is called the general solution of the equation. Hence the one-parameter family in Eq. (13) constitutes all the solutions of $y' + p(x)\, y = f(x)$ and is called the general solution.

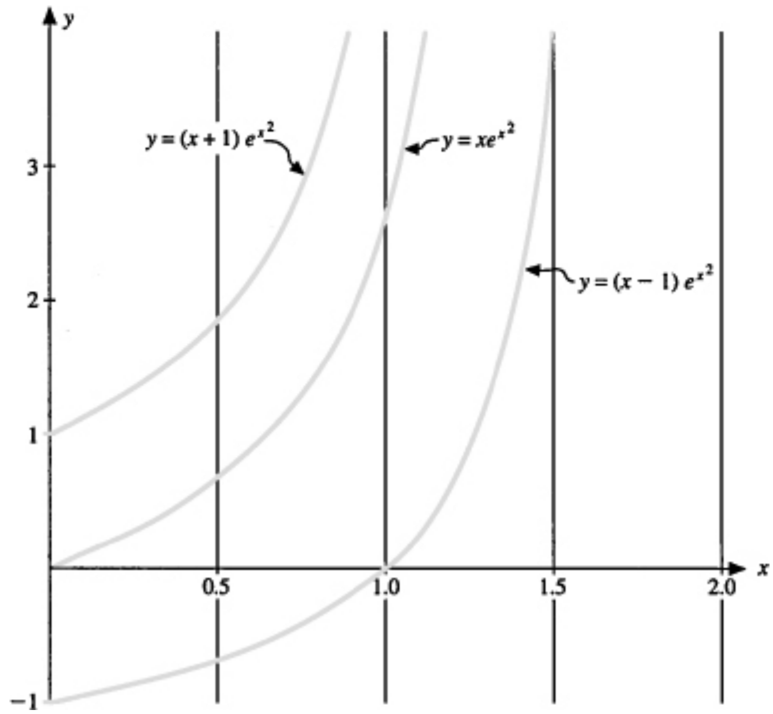
Example 1

Integrating Factor Method

Find the general solution of

$$y' - 2xy = e^{x^2} \qquad (-\infty < x < \infty) \qquad (14)$$

Figure 2.1 The one-parameter family of curves $y = (x + c)e^{x^2}$, where c is an arbitrary constant, represents the entire collection of solutions of $y' - 2xy = e^{x^2}$



Solution

Here $p(x) = -2x$, and so the integrating factor is

$$\mu(x) = e^{-\int 2x\,dx} = e^{-x^2} \tag{15}$$

Multiplying each side of the differential equation by the integrating factor, we get

$$e^{-x^2}(y' - 2xy) = 1$$

which can be rewritten as

$$\frac{d}{dx}(e^{-x^2}y(x)) = 1$$

Integrating, we get

$$e^{-x^2}y(x) = x + c$$

Solving for y, we find the solutions

$$y(x) = (x + c)\,e^{x^2} \tag{16}$$

where c is an arbitrary constant. A few of these solutions are drawn in Figure 2.1. ■

INITIAL-VALUE PROBLEM FOR FIRST-ORDER EQUATIONS

We are often interested in finding the single solution of a first-order equation that passes through a given point (x_0, y_0). This is the initial-value problem for first-order equations

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned} \tag{17}$$

The strategy for solving this problem is first to find all the solutions of the differential equation and then to determine which solution satisfies the initial condition $y(x_0) = y_0$. The following example illustrates this idea.

Example 2

Initial-Value Problem

Solve the initial value problem

$$\begin{aligned} \frac{dy}{dx} + \frac{3}{x}y &= \frac{\sin x}{x^3} \qquad (x > 0) \\ y(\pi/2) &= 1 \end{aligned} \tag{18}$$

Solution

Note that the differential equation is not defined when $x = 0$, and so we restrict the equation to the interval $(0, \infty)$, over which the coefficient $3/x$ and the nonhomogeneous term $(\sin x)/x^3$ are continuous. To solve this problem, we first find the general solution of the differential equation using the integrating factor method. Since $p(x) = 3/x$, the integrating factor is

$$\begin{aligned} \mu(x) &= e^{\int (3/x)\,dx} \\ &= e^{3 \ln x} \\ &= e^{\ln x^3} \\ &= x^3 \end{aligned} \tag{19}$$

Multiplying by $\mu(x)$ gives

$$\frac{d}{dx}(x^3 y) = \sin x$$

and by direct integration we find

$$x^3\, y(x) = -\cos x + c$$

Finally, dividing by x^3 gives the general solution

$$y(x) = \frac{c}{x^3} - \frac{\cos x}{x^3} \qquad (x > 0) \tag{20}$$

To determine which curve passes through the initial point ($\pi/2$, 1), we simply solve the equation $y(\pi/2) = 1$ for c. Doing this gives

$$1 = \frac{c}{(\pi/2)^3} - \frac{\cos(\pi/2)}{(\pi/2)^3}$$

or

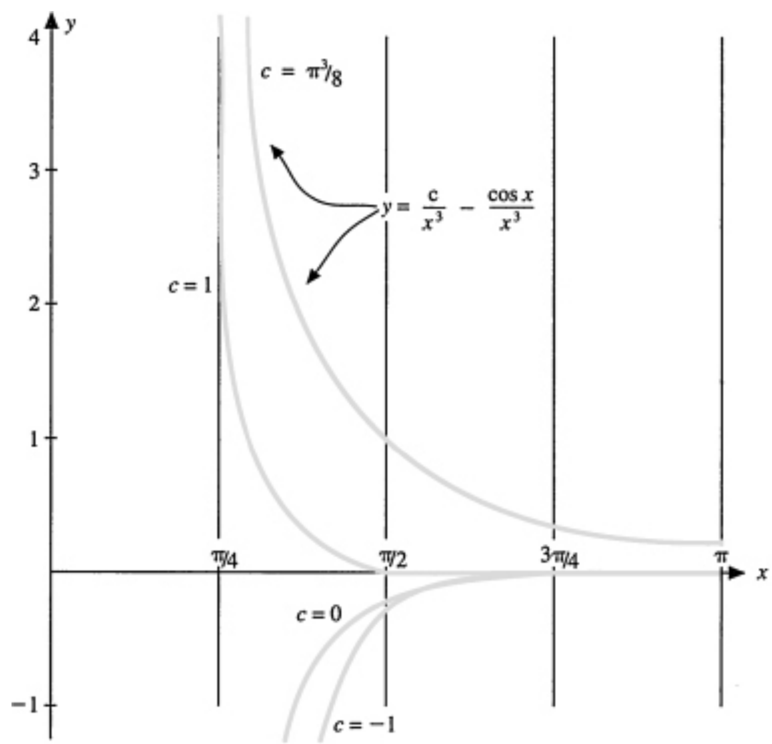
$$c = \frac{\pi^3}{8}$$

Hence the solution of the initial-value problem is

$$y(x) = \frac{\pi^3 - 8\cos x}{8x^3} \qquad (x > 0) \tag{21}$$

The graph of this solution is shown in Figure 2.2.

Figure 2.2 Solution of the initial-value problem



PROBLEMS: Section 2.1

For Problems 1–15, find the general solution to the indicated equation.

1. $\frac{dy}{dx} + 2y = 0$

2. $\frac{dy}{dx} + 2y = 3e^x$

3. $\frac{dy}{dx} - y = e^{3x}$

4. $\frac{dy}{dx} + y = \sin x$

5. $\frac{dy}{dx} + y = \frac{1}{1 + e^{2x}}$

6. $\frac{dy}{dx} + 2xy = x$

7. $\frac{dy}{dx} + 3x^2y = x^2$

8. $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}$

9. $x\frac{dy}{dx} + y = 2x$

10. $\cos x \frac{dy}{dx} + y \sin x = 1$

11. $\frac{dy}{dx} - \frac{2y}{x} = x^2 \cos x$

12. $\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3} \ (x \neq 0)$

13. $(1 + e^x)\frac{dy}{dx} + e^x y = 0$

14. $(x^2 + 9)\frac{dy}{dx} + xy = 0$

15. $\frac{dy}{dx} + \left(\frac{2x + 1}{x}\right)y = e^{-2x}$

For Problems 16–20, find the solution of the given initial-value problem.

16. $\frac{dy}{dx} - y = 1 \qquad y(0) = 1$

17. $\frac{dy}{dx} + 2xy = x^3 \qquad y(1) = 1$

18. $\frac{dy}{dx} - \frac{3}{x}y = x^3 \qquad y(1) = 4$

19. $\frac{dy}{dx} + 2xy = x \qquad y(0) = 1$

20. $(1 + e^x)\frac{dy}{dx} + e^x y = 0 \quad y(0) = 1$

21. **The Integrating Factor Identity** Verify the fundamental integrating factor identity

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}\,y(x)\right) = e^{\int p(x)\,dx}\left(\frac{dy}{dx} + p(x)y\right) \tag{22}$$

22. **Interchanging x and y to Get Linearity** Solve the non-linear differential equation

$$\frac{dy}{dx} = \frac{1}{x + y} \qquad y(-1) = 0$$

by considering the inverse function and writing x as a function of y. *Hint*: Using the basic identity from calculus that $dy/dx = \frac{1}{dx/dy}$, rewrite the given equation $y' = f(x, y)$ as $dx/dy = 1/f(x, y)$.

23. **A Tough Problem Made Easy** The differential equation

$$\frac{dy}{dx} = \frac{y^2}{e^y - 2xy}$$

would seem to be impossible to solve. However, if one treats y as the independent variable and x as the dependent variable and uses the relationship $dy/dx = \frac{1}{dx/dy}$, one can find an implicit solution. Find this implicit solution.

24. **Use of Transformations** Often a difficult problem is quite easy if it is viewed in the proper perspective.

(a)Solve the nonlinear equation

$$\frac{dy}{dx} + ay = by \ln y$$

where a and b are constants and can be solved by transforming the dependent variable* to $z = \text{In } y$.

(b)Use the result from part (a) to solve

$$\frac{dy}{dx} + y = y \ln y$$

25. **Bernoulli Equation** An equation that is not linear but can be transformed into a linear equation is the *Bernoulli equation*

$$\frac{dy}{dx} + p(x)y = q(x)y^n \qquad (n = 2, 3, 4, \ldots) \qquad (23)$$

(a)Show that the transformation $v = y^{1-n}$ reduces the Bernoulli equation to a linear equation in v.

(b) Use the transformation in part (a) to solve the Bernoulli equation

$$y' - y = y^3$$

26. **The Riccati Equation** The equation

$$\frac{dy}{dx} = a(x) + b(x)y + c(x)y^2 \tag{24}$$

is known as the *Riccati equation*.

(a)Show that if one solution $y_1(x)$ of the Riccati equation is known, then a more general solution containing an arbitrary constant can be found by making the substitution

$$y = y_1(x) + \frac{1}{v(x)} \tag{25}$$

and showing that $v(x)$ satisfies the linear equation

$$\frac{dv}{dx} = -(b + 2cy_1)\, v - c$$

(b)Verify that $y_1(x) = 1$ satisfies the Riccati equation

$$y' = -1 + 2y - y^2$$

and use this fact to find the general solution.

27. General Theory of First-Order Equations Show that if y_1 and y_2 are two different solutions of

$$\frac{dy}{dx} + p(x)y = f(x) \tag{26}$$

then $y_1, -y_2$ is a solution of the homogeneous equation

$$\frac{dy}{dx} + p(x)y = 0$$

28. Discontinuous Coefficients There are phenomena in which the coefficient $p(x)$ in the linear first-order equation is not continuous but has jump discontinuities. However, it is often possible to solve these types of problems with a little effort. Consider the initial-value problem

$$\frac{dy}{dx} + p(x)y = 1 \qquad y(0) = 0$$

where

$$p(x) = \begin{cases} 1 & (0 \le x \le 1) \\ 2 & (1 < x) \end{cases}$$

(a)Find the solution of the initial value problem in the interval $0 \le x \le 1$.

(b)Find the solution of the problem for $1 < x$.

(c)Sketch the graph of the solution for $0 \le x \le 4$.

29. Discontinuous Right-Hand Side Often the right-hand side $f(x)$ of the first-order linear equation is not continuous but has jump discontinuities. It is still possible to solve problems of this type with a little effort. For example, consider the problem

$$\frac{dy}{dx} + y = f(x) \qquad y(0) = 0$$

where

$$f(x) = \begin{cases} 1 & (0 \le x \le 1) \\ 0 & (1 < x) \end{cases}$$

(a)Find the solution of the initial value problem in the interval $0 \le x \le 1$.

(b)Find the solution of the problem for $1 < x$.

(c)Sketch the graph of the solution for $0 \le x \le 4$.

30. Comparing a Linear and Nonlinear Equation

(a)Verify the solutions:

$$\begin{array}{llll} y' = y & \Rightarrow y = e^x & & \text{(linear equation)} \\ y' = -y^2 & \Rightarrow y = x^{-1} & (x \neq 0) & \text{(nonlinear equation)} \end{array}$$

(b)Verify

$$\begin{array}{lll} y' = y & \Rightarrow \quad y = ce^x & \text{for all } c \\ y' = y^2 & \Rightarrow \quad y = cx^{-1} & \text{for only } c = 0, -1 \end{array}$$

31. Error Function Express the solution of $y' = 1 + 2xy$ in terms of the **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \qquad (27)$$

32. Computer Problem–Sketching Solutions Use a graphing calculator or a graphing package for your computer* to plot some of the solutions of the one-parameter family of solutions drawn in Figure 2.1. You might try sketching some solutions for different values of the parameter c than have been drawn in the text. The author used the computer package *MICRO CALC* to draw the curves shown in Figure 2.1.

33. Computer Problem–Sketching a Solution Redraw the solution of the initial-value problem drawn in Figure 2.2 using either a graphing calculator or a computer.

34. Today's Journal Entry Spend ten minutes exploring your thoughts about some aspect of the integrating factor. Is it possible to solve *algebraic equations* using an integrating factor? Were you disappointed in the complicated-looking solution to the first-order equation? What did you expect? Will the integrating factor method always work? Can you think of another way to solve the first-order equation? Do you think this course is going to be worthwhile? Date your entry.

2.2 SEPARABLE EQUATIONS

PURPOSE

To solve the class of first-order differential equations of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

known as separable equations by a method known as separation of variables. The importance of this class of equations lies in the fact that many important nonlinear equations are separable and hence solvable.

SOLVING SEPARABLE EQUATIONS

The very simplest differential equation is the one studied in calculus,

$$\frac{dy}{dx} = f(x) \tag{1}$$

where $f(x)$ is a given continuous function. In calculus we learned that we can solve this equation by essentially "integrating both sides" of the equation, getting

$$y(x) = \int f(x)\, dx + c \tag{2}$$

where c is an arbitrary constant and the integral sign denotes any single antiderivative of $f(x)$. We now see that this procedure can be applied to a broader class of differential equations, known as **separable equations**, having the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \tag{3}$$

Clearly, any separable equation* reduces to the simpler form in Eq. (1) when we have $g(y) = 1$. To solve a separable differential equation, we rewrite it as