Method of Characteristics

Consider $u(x,t)$

$$u_t + c(x,t,u)\,u_x = g$$

1. Find characteristic curves $(x(t),t)$ passing through $(k,0)$, solving

$$\begin{cases} \dfrac{dx}{dt} = c(x(t), t, u(x(t),t)) \\ x(0) = k \end{cases}$$

2. Solve the <u>ODE</u> along characteristics

$$\frac{d}{dt}\left[u(x(t),t)\right] = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + u_t$$

$$= u_x \cdot c + u_t$$

$$\left(\text{since } u \text{ solves the PDE}\right) \quad = g$$

Solving $\dfrac{du}{dt} = g$ yields solutions along characteristics.

3. Reconstruct the solution using an initial condition $u(x,0) = u_0(x)$.

Ex 1 Transport

$$\begin{cases} u_t + c u_x = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad \boxed{c \text{ constant}}$$

Char. ODE (IVP)

$$\begin{cases} \dfrac{dx}{dt} = c \\ x(0) = k \end{cases} \longrightarrow \quad x(t) = ct + k \qquad \boxed{k = x - ct}$$

$$\frac{d}{dt}\left[u(x(t), t)\right] = u_x \frac{dx}{dt} + u_t$$

$$= u_x c + u_t$$

by the PDE $= 0$

Find solution to $\dfrac{d}{dt} u = 0 \ldots$

$u(x(t), t) = \text{constant} = C$

$u(x(0), 0) = C$

$\parallel$ ⟵ from the char. IVP

$u(k, 0)$ ⟵ from the IC

$\parallel$

$u_0(k)$

so $u(x(t), t) = u_0(k)$ but $k = x - ct$

Reconstruct: $u(x,t) = u_0(x - ct)$

Ex 2 Nonconstant Coeff.

$$\begin{cases} u_t + x^2 u_x = 0 \\ u(x,0) = e^x \end{cases}$$

Char. ODE/IVP

$$\begin{cases} \dfrac{dx}{dt} = x^2 \\ x(0) = k \end{cases}$$

$$\frac{dx}{x^2} = dt \Rightarrow -\frac{1}{x} = t + C$$

plug in $x(0) = k$

$$-\frac{1}{k} = C$$

So characteristics given by

$$-\frac{1}{x} = t - \frac{1}{k} \quad \text{or} \quad k = \frac{1}{t + 1/x}$$

Since RHS of PDE is zero

$u(x(t), t) = $ const. along characteristic curves $k = \dfrac{1}{t + 1/x}$

$$\begin{aligned} \text{So} \quad u(x(t), t) &= u(x(0), 0) \\ &= u(k, 0) \\ &= e^k = \exp\left(\frac{1}{t + 1/x}\right) \end{aligned}$$

Example 3 Inhomogeneous

$u(x,y)$

$$\begin{cases} u_y + u_x = 1 \\ u(x,0) = \sin(x) \end{cases}$$

Char ODE

$$\begin{cases} \frac{dx}{dy} = 1 \\ x(0) = k \end{cases}$$

Note: constant coeffs. will yield linear characteristics.

$$\boxed{k = x - y}$$

Solve the ODE along characteristics

$$\frac{d}{dy}\left[u(x(y),y)\right] = u_x \frac{dx}{dy} + u_y$$

$$= u_x + u_y$$

By the PDE $\longrightarrow$ $= 1$

$$\frac{du}{dy} = 1 \Rightarrow u(x(y),y) = y + C$$

Reconstruct:

$$u(x(0),0) = 0 + C \Rightarrow u(k,0) = C$$

$$\Rightarrow \sin(k) = C$$

So $u(x,y) = y + \sin(x-y)$

Ex 4

$$\begin{cases} \theta_t + 2\theta_x = t + \theta, & \theta(x,0) = \theta_0(x) \end{cases}$$

characteristics: $\kappa = x - 2t$

ODE

$$\frac{d}{dt}\left[\theta(x(t),t)\right] = t + \theta$$

$$\hookrightarrow \quad \theta' - \theta = t$$

~~solve using warm up:~~

$$\theta(x(t),t) = -(1+t) + Ce^{t}$$

$\boxed{t=0}$

$$\theta(x(0),0) = -(1+0) + Ce^{0}$$

$$\theta(\kappa,0) = -(1+0) + C$$

$$\Rightarrow \quad C = \theta(\kappa,0) + 1$$

$$\theta(x,t) = -(1+x) + \left[\theta(\kappa,0) + 1\right]e^{t}$$

$\kappa = x - 2t$

$$= -(1+x) + \left[\theta_0(x-2t) + 1\right]e^{t}$$