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HeatWave Fact Sheet

MATH 404, Fall 2025

These notes are meant to be a concise guide to the two principal second order evolutions we have studied.

1 Wave Equation

1.1 Basic Facts

The homogeneous wave equation is given by $u_{tt} - c^2 u_{xx} = 0$, where $u(x,t)$ is the solution variable giving the transverse displacement of a infinite length, thin wire with tension at position x and time t. The value c is the *wave speed*.

The general solution (with no reference to initial data) is given by

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F, G are arbitrary, scalar-valued functions of one argument. (This represents a fixed right moving profile and a fixed left moving profile.) This can be obtained via *operator factoring*.

The Cauchy problem for the wave equation on $\mathbb{R}$ is

$$u_{tt} - c^2 u_{xx} = 0; \quad u(x,0) = \phi(x); \quad u_t(x,0) = \psi(x),$$

where we think of $\phi(x)$ as the initial wave profile, and $\psi(x)$ as the initial velocity profile.

The *d'Alembert* solution to the Wave Cauchy problem is written in terms of the initial data:

$$u(x,t) = \frac{1}{2}\Big\{\phi(x+ct) + \phi(x-ct)\Big\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

For *finite energy initial data* ($\phi', \psi \in L^2(\mathbb{R})$), this solution is unique (in the sense of energies).

The *energy* for the wave equation is given by

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t(x,t))^2 + c^2(u_x(x,t))^2 dx,$$

with clear *potential* and *kinetic* components. If the initial data are of finite energy, then any (weak) solution has the property that

$$E(t) = E(0) = E(\phi,\psi), \quad \forall\ t \geq 0.$$

1.2 General Comments

Helpful facts:

- The linear operator $\partial_t^2 - (c\partial_x)^2 = (\partial_t + c\partial_x)(\partial_t - c\partial_x)$. This factorizations shows how the wave equation involves right-moving and left-moving transport of information.
- The two principal characteristics associated to an initial space-time value $(x_0, 0)$ are: $x_0 = x - ct$, $x_0 = x + ct$. The first is a line in the x-t plane of slope $1/c$ intersecting the point $(x_0, 0)$ and the second is the same, but with slope $-1/c$.

We think of wave dynamics in terms of signal propagation on $\mathbb{R}$, where the initial "disturbance" is measured in terms of the initial data ϕ and ψ.

The first term in the d'Alembert solution is contributed from the initial condition ϕ, whereas the integral term is contributed by the initial velocity ψ. We can think of the solution as being constructed via the superposition of these two effects.

At any time t, the solution is constructed from the values of the initial data on the set $[x - ct, x + ct]$. Information outside of this set does not affect the value of $u(x,t)$.

From the above, it is clear that information (from the initial conditions) does not propagate faster than the speed c. When $\psi \equiv 0$, *information propagates at exactly the speed c*. This is known as the principle of *causality*.

The d'Alembert solution propagates the regularity of the initial conditions—thus discontinuities are transported along characteristics. The solution *maintains the spatial regularity of the initial data*.

We can consider lower order terms: $u_{tt} - c^2 u_{xx} + du_t + au_x + bu = 0$. The term involving $d > 0$ is known as "damping" or "dissipation", and when $d = a = 0$ with $b > 0$, the equation is referred to as the Klein-Gordon (dispersive) equation. Through a variable substitution, one can reduce any equation with lower order terms to the Klein-Gordon equation.

2 Heat/Diffusion Equation

2.1 Basic Facts

The homogeneous heat/diffusion equation is given by $u_t - Du_{xx} = 0$, where $u(x,t)$ is the solution variable giving the "quantity" (temperature, energy, concentration) of a substance at x and time t.

A general solution (with no reference to initial data) is given by

$$u(x,t) = c_1 \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-s^2} ds + c_0,$$

where c_1, c_0 are arbitrary constants. This is obtained through the Boltzmann transformation: $z = \dfrac{x}{\sqrt{Dt}}$.

The Cauchy problem for the heat equation on $\mathbb{R}$ is

$$u_t - Du_{xx} = 0; \quad u(x,0) = \phi(x),$$

where we think of $\phi(x)$ as the initial concentation profile.

For the special initial data $\phi(x) = H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ the solution to the Cauchy problem is

$$h(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right].$$

The *fundamental solution* to the heat equation (also known as the *Green's Function* or *heat kernel*) is

$$S(x,t) = \frac{1}{2\sqrt{\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right).$$

We interpret this function to have initial condition $\phi(x) = \delta(x)$, of course remembering that the δ "function" is not a traditional function. We note the relationship that $\partial_x[h(x,t)] = S(x,t)$, which is consistent with the "fact" that $\partial_x(H(x)) = \delta(x)$ (in the sense of distributions).

The *convolution* solution to the general Heat/Diffusion Cauchy problem is written in terms of convolution of the initial data with the heat kernel:

$$
\begin{aligned}
u(x,t) =& \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} \phi(y) \exp\left(\frac{-(x-y)^2}{4Dt}\right) dy \\
=& \int_{-\infty}^{\infty} \phi(y) S(x-y,t) dy = \phi(x) * S(x,t) = S(x,t) * \phi(x).
\end{aligned}
$$

The initial condition is satisfied (in general) in the sense that

$$\lim_{t \searrow 0} u(x,t) = \phi(x).$$

For finite energy initial data, this solution is unique (in the sense of energies). (In fact, solutions are unique, even if the initial data has polynomial-like growth at infinity.)

The energy for the heat equation is given by

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u(x,t))^2 dx.$$

If the initial data is of finite energy, then the (energy) solution has the property that

$$E(t) + D \int_0^t \int_{-\infty}^{\infty} (u_x(x,\tau))^2 dx d\tau = E(0) = E(\phi), \quad \forall\ t > 0.$$

2.2 General Comments

Helpful Facts:

- The heat kernel $S(x,t)$ is a the density function of a normal random variable with mean $\mu = 0$ and $\sigma = \sqrt{2Dt}$. As t increases from 0, the maximal amplitude of $S(x,t)$ decreases, and the variance (spread of mass) of the density increases.
- The solution $u(x,t) = \frac{1}{2}\left[1 + \text{erf}(x/2Dt)\right]$ for initial condition $\phi(x) = H(x)$ is the distribution function of the above random variable.
- For all $t > 0$, we have $\displaystyle\int_{-\infty}^{\infty} S(x,t) dx = 1$.
- In reference to the δ function, we think of $\lim_{t \searrow 0} S(x,t) = \delta(x)$. Recall the defining properties of the δ "function":

  $$\begin{cases} \delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases} \\ \displaystyle\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0), \quad \forall\ f(x). \end{cases}$$

  In particular: $\displaystyle\int_{-\infty}^{\infty} \delta(x) dx = 1$.
  (It is not really a traditional function, but should can be thought of as a measure, generalized function, or distribution.)
- The *convolution* of two functions f, g defined on $\mathbb{R}$ is given by

  $$f(x) * g(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x) g(x-y) dy = \int_{-\infty}^{\infty} f(x-y) g(x) dy.$$

  The convolution can be thought of as a weighted average, with the graph of f being weighted by g, or vice versa.

In general, computing the solution explicitly using the convolution representation cannot be done directly. In some special cases it can, otherwise, numerical techniques are used.

If $u(x, t)$ is a solution to the heat equation on $\mathbb{R}$, we note that:

- $u(x - y, t)$ is a solution for any $y \in \mathbb{R}$. By superposition, so, then, is $\displaystyle\int_{-\infty}^{\infty} \phi(y) u(x - y, t) dy$.
- any derivative of $u(x, t)$ (e.g., $u_x, u_t, u_{xx}, u_{tt}, ...$) is also a solution.
- for any $\alpha > 0$, we have that $u(\alpha x, \alpha^2 t)$ is a solution.

As per the above, the best way to think about *convolution* here is in the context of a general superposition of solutions, though the concept is much broader. A solution for the heat Cauchy problem is simply a convolution of the fundamental solution with the initial condition.

We may think of heat/diffusion dynamics as the initial distribution being "spread out" (in time) according to the diffusivity constant D. On a point-by-point basis x, the solution decays—heat is "spread out" to infinity.

We think of the convolution solution as a weighted average according to the weights $S(x - y, t)$: the value of $u(x, t)$ is a weighted average of the initial values of ϕ *near* x—the Gaussian $S(x - y, t)$ places a large emphasis on the y near x, and decays rapidly as y ventures farther from x. As t increases, the heat kernel spreads out, thus this effect is minimized.

We note that at a time t, the solution $u(x, t)$ depends on *all of the information* $\phi(x)$ (in terms of the convolution integral). We regard this as *infinite speed of propagation*, since any change to the initial data is immediately "felt" by the solution.

As $t \searrow 0$, we think of $S(x, t)$ as being approximations of the δ functions; in this way we can use the properties of the δ function to recover the initial condition.

In general, solutions to the homogeneous heat equation on $\mathbb{R}$ *decay*. From the derivation of the PDE, we note that that the *flux*$= -cu_x$, and thus energy/heat moves *down the gradient*. From the energy identity, the rate of dissipation is given in terms of $[u_x]^2$. From the PDE, we see that large u_{xx} values (the profile has large magnitude concavity) instigate large time changes in the solution u_t. Regions with large gradients ($|u_x| >> 1$) experience large fluxes and regions with large concavity ($|u_{xx}| >> 1$) experience rapid decay.

We did not explicitly prove this, but solutions to the heat equation are very smooth—namely, can be differentiated arbitrarily, regardless of whether the initial data $\phi(x)$ is even continuous. If the initial data $\phi(x)$ is smooth and of compact support, then the solution is also smooth, with decay of all derivatives at $\pm\infty$. If the initial data $\phi(x)$ is not smooth, it is immediately "smoothed out" by the solution.

We can consider lower order terms: $u_t - Du_{xx} + au_x + bu = 0$. The term involving $b > 0$ is a form of "damping", and when the term involving a is an *advection* (or transport) term. Through a variable substitution, one can reduce any equation with lower order terms to the heat equation.

3 Short Compare/Contrast

- **Wave** Equation: hyperbolic, finite speed of propagation, information transported, regularity preserving, conserved energy, time reversible
- **Heat** Equation: parabolic, infinite speed of propagation, information lost in time, smoothing of solutions, energy dissipated, time irreversible