Let's use $\mathcal{L}$ to solve an IBVP.

I'll do Dirichlet wave, inhomogeneous.

I'll help, but you'll do inhomogeneous heat.

Consider the inhomog. WEQ on $[0,\infty)$

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & x>0,\ t>0 \\ u(0,t) = g(t) & t \geq 0 \quad \text{(compatibility)} \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) & x \geq 0 \end{cases}$$

By superposition, we need only solve the sub-problem

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & x>0,\ t>0 \\ w(0,t) = g(t) & t \geq 0 \\ w(x,0) = w_t(x,0) = 0 & x \geq 0 \end{cases}$$

We'll use $\mathcal{L}$ to eliminate t.

For $w(x,t)$, write $\mathcal{L}[w(x,t)](s) = W(x,s)$

Strategy

$$
\begin{tikzcd}
\text{IBVP in } x + t \arrow[rr, bend left, "\mathcal{L}"] \arrow[d, dashed, "\text{solving indirectly}"] & & \text{ODE in } x, s \arrow[d] \\
\begin{array}{c} u(x,t) \\ \text{solves IBVP} \end{array} & & \begin{array}{c} \text{solve ODE} \\ U(x,s) \end{array} \arrow[ll, bend left, "\mathcal{L}^{-1}"]
\end{tikzcd}
$$

Take $\mathcal{L}$ of the WEQ:

$$\mathcal{L}[w_{tt}(x,t)](x,s) - c^2 \mathcal{L}[w_{xx}(x,t)](x,s) = 0$$

$$s^2 W(x,s) - s\,\cancelto{0}{w(x,0)} - \cancelto{0}{w_t(x,0)} - c^2 W_{xx}(x,s) = 0$$

$$\Rightarrow \quad W_{xx}(x,s) - \frac{s^2}{c^2} W(x,s) = 0$$

AN ODE in x! Solve in x w/ s constant.

$$W(x,s) = c_1 e^{-s/c\, x} + c_2 e^{s/c\, x}$$

$$W(x,s) = c_1 e^{-s/c\,x} + \cancel{c_2 e^{s/c\,x}} \to 0$$

unbounded for $x > 0$.

$$W(x,s) = c_1 e^{-s/c\,x}$$

so $W(0,s) = c_1 \Rightarrow$ By the BC $W(0,s) = \mathcal{L}[g(t)] = G(s)$

So in the dual domain we know $W(x,s)$:

$$W(x,s) = G(s)e^{-s/c\,x}$$

So we need to invert this from the table.

27) $H(t-t_0)f(t-t_0) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} e^{-t_0 s}F(s)$

So
$$\underset{\displaystyle w(x,t)}{\overset{\displaystyle \mathcal{L}^{-1}[W(x,s)](x,t)}{\|}} = \mathcal{L}^{-1}\left[G(s)e^{-s/c\,x}\right](x,t) = g\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)$$

Note: the sol^{n} only makes reference to the data $g(t)$ at $x=0$.

Consider on $x \in [0,\infty)$ the HEQ:

$$\begin{cases} u_t - D u_{xx} = F(x,t) & x>0,\ t>0 \\ u(0,t) = g(t) & t \geq 0 \\ u(x,0) = \varphi(x) & x \in [0,\infty) \end{cases}$$

compatinity

By <u>superposition</u> we only need to learn how to solve this subproblem:

$$\begin{cases} w_t - D w_{xx} = 0 & x>0,\ t>0 \\ w(0,t) = g(t) & t \geq 0 \\ w(x,0) = 0 & x \in [0,\infty) \end{cases}$$

We'll use $\mathcal{L}$ to eliminate t.

$$\mathcal{L}\left[w_t - D w_{xx} = 0\right]$$

0 since $\varphi = 0$ here

$$\Rightarrow \left[sW(x,s) - s\,\cancel{w(x,0)}\right] - D W_{xx}(x,s) = 0$$

$$\Rightarrow W_{xx}(x,s) - \frac{s}{D} W(x,s) = 0$$

<u>Solve</u>

$$W(x,s) = C_1 e^{\sqrt{s}/\sqrt{D}\,x} + C_2 e^{-\sqrt{s}/\sqrt{D}\,x}$$