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Second Order PDE Solution Methods for Unbounded Domains

MATH 404, Spring 2023

These notes are meant to be a concise guide to solution methods for second order equations.

1 Problems with $x \in \mathbb{R}$

1.1 Solutions

For the **homogeneous heat/diffusion Cauchy problem**

$$\begin{cases} u_t - Du_{xx} = 0, & x \in \mathbb{R}, \quad t \in (0,\infty) \\ u(x,0) = \phi(x), & t = 0 \end{cases}$$

we have the convolution heat kernel solution

$$u(x,t) = \phi(x) * S(x,t) = \int_{-\infty}^{\infty} \phi(y) S(x-y,t)dy,$$

where

$$S(x,t) = \frac{1}{2\sqrt{Dt}} e^{-\frac{x^2}{4Dt}}$$

is the standard heat kernel and $*$ refers to convolution.

For the **homogeneous wave Cauchy problem**

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R}, \quad t \in (0,\infty) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & t = 0 \end{cases}$$

we have the d'Alembert solution

$$u(x,t) = \frac{1}{2}\left\{\phi(x+ct) + \phi(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

1.2 Superposition and Duhamel

For the inhomogeneous heat equation Cauchy problem

$$\begin{cases} u_t - Du_{xx} = F(x,t), & x \in \mathbb{R}, \quad t \in (0,\infty) \\ u(x,0) = \phi(x), & t = 0 \end{cases}$$

we use the **principle of superposition** and Duhamel's principle.

If $u_1(x,t)$ solves $u_t - Du_{xx} = F(x,t),\ u(x,0) = 0$, and $u_2(x,t)$ solves $u_t - Du_{xx} = 0, \quad u(x,0) = \phi(x)$, then $u = u_1 + u_2$ solves to the inhomogeneous Cauchy problem.

To solve for u_1 we use **Duhamel's principle**. Suppose that $w = w(x,t;\tau)$ is a solution to the τ-parametrized heat equation

$$\begin{cases} w_t - Dw_{xx} = 0 \\ w(x,0;\tau) = F(x,\tau). \end{cases} \quad .$$

(Note, one can obtain a solution to this via the convolution heat kernel approach.)

Then the function $u_1(x,t) = \int_0^t w(x, t-\tau; \tau) d\tau$ is a solution to

$$u_t - Du_{xx} = F(x,t), \quad u(x,0) = 0.$$

For the inhomogeneous wave equation Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & x \in \mathbb{R}, \quad t \in (0,\infty) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & t = 0 \end{cases}$$

we use the **principle of superposition** and Duhamel's principle.

If $u_1(x,t)$ solves $u_{tt} - c^2 u_{xx} = F(x,t)$, $u(x,0) = u_t(x,0) = 0$, and $u_2(x,t)$ solves $u_{tt} - c^2 u_{xx} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, then $u = u_1 + u_2$ solves to the inhomogeneous Cauchy problem.

To solve for u_1 we use **Duhamel's principle**. Suppose that $w = w(x,t;\tau)$ is a solution to the τ-parametrized wave equation

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0;\tau) = 0, \quad w_t(x,0;\tau) = F(x,\tau) \end{cases}.$$

(Note, one can obtain a solution to this wave equation using d'Alembert's approach.)

Then the function $u_1(x,t) = \int_0^t w(x, t-\tau; \tau) d\tau$ is a solution to

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad u(x,0) = u_t(x,0) = 0.$$

1.3 Fourier Transform Method

In what follows, $x \in \mathbb{R}$ is a spatial variable that will be transformed; y is taken to be the general non-transform variable (which could, for evolutions, be t).

Consider the problem with Cauchy data $\phi(x)$ (or, for the wave equation, $\phi(x), \psi(x)$). Take the Fourier transform in x ($\mathscr{F}_x[\cdot]$) of the entire problem, and use the transformed data $\hat{\phi}(\xi)$ in the resulting ODE in t. You may need to invoke an auxiliary condition as well. After solving for $\hat{u}(\xi,t)$, invert the Fourier transform ($\mathscr{F}_x^{-1}[\cdot]$) to obtain a solution.

$$\begin{array}{ccc} \text{PDE (in } x, y) & \xrightarrow{\mathscr{F}_x} & \text{ODE (in } y) \\ \Big\downarrow & & \Big\downarrow \text{solve} \\ \text{solution } \; u(x,y) & \xleftarrow{\mathscr{F}_x^{-1}} & \hat{u}(\xi, y) \end{array}$$

2 Problems with $x \in [0, \infty)$

2.1 Superposition for Boundary Data

Consider a general second order problem in $u(x,t)$ with Cauchy data $\phi(x)$ (or, for the wave equation, $\phi(x), \psi(x)$), and boundary data at $x = 0$ given by $g(t)$ (this can be Dirichlet or Neumann data). We invoke the principle of superposition: If

- [**Subproblem A**] If u_1 solves the IBVP with given Cauchy data and homogeneous boundary data $g(t) \equiv 0$
- [**Subproblem B**] If u_2 solves the IBVP with zero Cauchy data and given boundary data $g(t)$
- Then $u = u_1 + u_2$ will solve the full IBVP with both types of data given.
- This can be extended, in a natural way, to include inhomogeneous forcing $F(x,t)$, as above.

2.2 Reflection Principle

For a given function $f(x)$ defined on $(0, \infty)$, let us consider the odd extension $f_O(x)$ and the even extension $f_E(x)$ defined on $\mathbb{R}$ as:

$$f_0(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

$$f_E(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

The **reflection principle** is invoked to solve *Subproblem A* above by utilizing odd extension(s) for homogeneous Dirichlet boundary conditions and even extension(s) for homogeneous Neumann boundary conditions.

For homogeneous Dirichlet conditions:

- Extend the initial data $\phi(x)$ to obtain $\phi_O(x)$ on $\mathbb{R}$.
- Solve the Cauchy problem with Cauchy data $\phi_O(x)$.
- Note that the resulting solution $u_O(x,t)$ is defined for $x \in \mathbb{R}$ and satisfies the PDE for all $x > 0$.
- Note that $u_O(0,t) = 0$ for all $t \geq 0$.
- Note that $u_O(x,0) = \phi(x)$ for all $x > 0$.
- Note that $u_O(x,t)$, when **restricted** to $x > 0$, satisfies the initial and boundary conditions, as well as the PDE for $x > 0$, and is thus the sought after solution.

For homogeneous Neumann conditions:

- Extend the initial data $\phi(x)$ to obtain $\phi_E(x)$ on $\mathbb{R}$.
- Solve the Cauchy problem with Cauchy data $\phi_E(x)$.
- Note that the resulting solution $u_E(x,t)$ is defined for $x \in \mathbb{R}$ and satisfies the PDE for all $x > 0$.
- Note that $u_{E,x}(0,t) = 0$ for all $t \geq 0$.
- Note that $u_E(x,0) = \phi(x)$ for all $x > 0$.
- Note that $u_E(x,t)$, when **restricted** to $x > 0$, satisfies the initial and boundary conditions, as well as the PDE for $x > 0$, and is thus the sought after solution.

One should make the appropriate changes if the problem is a pure BVP for $(x,y) \in \mathbb{H}$. Also note that if solving an IBVP for the wave equation, one should extend both $\phi(x)$ and $\psi(x)$ appropriately.

2.3 Laplace Transform Method

In what follows, $t \in [0, \infty)$ is the time-like variable that will be transformed; x is taken to be the spatial, non-transform variable.

With the principle of superposition, consider the IBVP or BVP with zero Cauchy data, and boundary data given by $g(t)$. Take the Laplace transform in t ($\mathscr{L}_t[\cdot]$) of the entire problem (the zero Cauchy data will simplify this), and use the transformed data $G(s)$ in the resulting ODE in x. You may need to invoke an auxiliary condition as well. After solving for $U(x, s)$, invert the Laplace transform ($\mathscr{L}_t^{-1}$) to obtain a solution.

This is summarized

$$\begin{array}{ccc} \text{PDE (in } x, t) & \xrightarrow{\mathscr{L}_t} & \text{ODE (in } x) \\ \downarrow & & \downarrow \text{solve} \\ \text{solution } \; u(x,t) & \xleftarrow{\mathscr{L}_t^{-1}} & U(x,s) \end{array}$$