Solutions + Verifying; ODEs as ?s

Worst case scenario for nonlinear ODEs is (n^{th} order), $y(x)$:

$$G(x, y, y', y'', \dots, y^{(n)}) = f(x)$$

ex

$$\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2} = x^2$$

Better case (n^{th} order), $y(x)$:

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$$

semilinear

A sol^n to the above on $x \in (a,b)$ is any n-times differentiable function on (a,b) so that ODE relation is also satisfied on (a,b).

A solution can have parameters or general constants in it. (order, linearity)

ex: $\ddot{x} + x = 0$, $t \in [0, \infty)$ ✱

has sol^{n} $x(t) = c_1 \cos(t) + c_2 \sin(t)$

(this is an explicit representation).

Check: $\dfrac{dx}{dt} = -c_1 \sin(t) + c_2 \cos(t)$

$\dfrac{d^2x}{dt^2} = -c_1 \cos(t) - c_2 \sin(t)$

So $\ddot{x} + x = [-c_1 \cos(t) - c_2 \sin(t)]$
$+ [c_1 \cos(t) + c_2 \sin(t)]$

$= 0$ for all $t \in [0, \infty)$

and any $c_1, c_2 \in \mathbb{R} \equiv (-\infty, \infty)$.

So $x(t) = c_1 \cos(t) + c_2 \sin(t)$ is a sol^{n} to ✱ on $[0, \infty)$.

Note: Showing sol^{n} needs to be done this way.

For nonlinear DEs, things can be more complicated.

Let's try something fun.

Consider the <u>relation</u> for a circle

$$x^2 + y^2 = r^2 \qquad (r > 0, \; x \in (-r, r))$$

Let's differentiate implicitly:

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[r^2\right]$$

$$\Longleftrightarrow \quad 2x + 2yy' = 0$$

$$\Longleftrightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \qquad \text{nonlinear ODE}$$

So the <u>relation</u> $x^2 + y^2 = r^2$ solves the ODE on $x \in (-r, r)$.

Note: We can sometimes make relations explicit, but it's not always needed.

$$\left(y = (r^2 - x^2)^{1/2} \rightarrow y' = -x\,(r^2 - x^2)^{-1/2} \right)$$