Picard II: "Make it so."

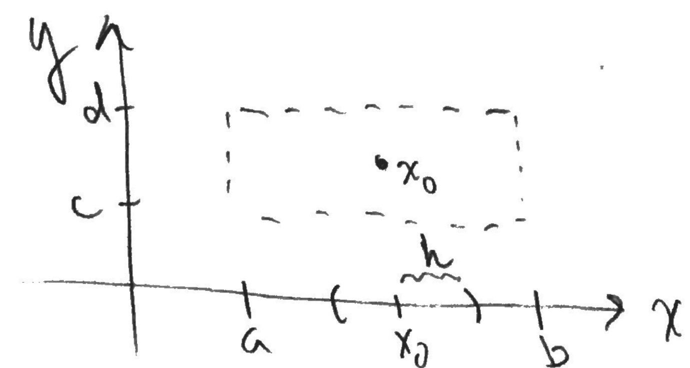
Here it is:

Picard's Theorem (for first order)

Consider $(*)\ \begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 . \end{cases}$

[Suppose] $f(x,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on some rectangle (open) $(a,b) \times (c,d)$ containing (x_0, y_0)

[Then] there exists a $\varphi(x)$ and $h > 0$ so that on $(x_0 - h, x_0 + h)$ $\varphi(x)$ is the unique differentiable function solving $*$.



How is it proved? That is, how do the hypotheses yield the conclusion?

properties of f in $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ $\longmapsto$ unique local sol^{n}. $y = \varphi(x)$.

Well, by an iterative algorithm.

Take the ODE and integrate from x_0 to $x \in (a,b)$: ξ

$$y(x) - y(x_0) = \int_{x_0}^{x} f(\xi, y(\xi))\,d\xi$$

Rewrite: $$y(x) = y(x_0) + \int_{x_0}^{x} f(\xi, y(\xi))\,d\xi$$

(Implicit)

Define $$\varphi_n(x) = y(x_0) + \int_{x_0}^{x} f(\xi, \varphi_{n-1}(\xi))\,d\xi$$

with $\varphi_0(x) = y_0$ $\leftarrow$ const. (yields a series)

The sequence $\{\varphi_n(x)\}$ converges to a function $y(x)$. A (fixed point) sol^{n}!

MATH 301 + 302

Check to see how the three examples from before can be interpreted in terms of Picard's

Sketches below (I will do a full write up Tues.)

Example 1 #41

$$\begin{cases} yy' = x^2 + y^2 \\ y(0) = 1 \end{cases}$$

Rewrite: $y' = \frac{x^2}{y} + y$

$y \neq 0$, but $y_0 = 1$

$f(x,y) = y^{-1}x^2 + y$ continuous for $y > 0$

$\frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + 1$

continuous for $y > 0$

So let $R = (-1,1) \times (1/2, 2)$ then Picard applies.

Example 2

$$\begin{cases} y' = -x/y \\ y(1) = \boxed{0} = y_0 \end{cases}$$

Need $y \neq 0$ so for $f(x,y) = -x/y$ there is no rectangle containing $(1,0)$ for which

$f(x,y) = -x/y$ is continuous. Hence the theorem cannot be applied. (Careful though.)