# Picard I: Motivational Examples



("Engage")

Given a DE or IVP, first order in $y$:

$$F(x, y, y') = 0$$

or

$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

- How do we know there is a sol$^{\underline{n}}$? (we shouldn't look for something that doesn't exist!)
- How many? One, a few, infinite? (If your DE model is for a bridge...)
- How "long" (on what interval) does the sol$^{\underline{n}}$ exist?

Can we <u>know</u> from properties of $f$, $f'$, $F$, etc. the answers above?

## (Silly) Example 1

$$F(x, y') = x^2 + (y')^2$$ ← perfectly nice $F$

ODE. A) $F(x, y') = -1$

B) $F(x, y') = 1$

A CANNOT HAVE A SOL$^{n}$. OH NO!

## Example 2 $\boxed{\text{IVP}}$

$$(*)\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}$$

$f(y) = y^{1/3}$

$(x \geq 0)$

Separable (next section)

$$\frac{dy}{dx} = y^{1/3} \iff y^{-1/3}\,dy = dx$$

Integrate:

$$\frac{3}{2} y^{2/3} = x + C$$

$$\iff \quad y(x) = \left(\frac{2}{3}x + C\right)^{3/2}$$

Impose $y(0) = 0 \Rightarrow C = 0$; $y(x) = \left(\frac{2}{3}\right)^{3/2} x^{3/2}$

$$y' = \frac{3}{2}\left(\frac{2}{3}\right)^{3/2} x^{1/2} = \left(\frac{2}{3}\right)^{1/2} x^{1/2}$$

$$y^{1/3} = \left(\left(\frac{2}{3}\right)^{3/2} x^{3/2}\right)^{1/3} = \left(\frac{2}{3}\right)^{1/2} x^{1/2}$$

But you can check that:

$$y_c(x) = \begin{cases} \left(\frac{2}{3}(x-c)\right)^{3/2} & x \geq c \\ 0 & x \in [0, c) \end{cases}$$

Also satisfies the IVP FOR ANY $c \in [0, \infty)$. So... infinite solutions to $*$. OH NO!

Example 3

$$\begin{cases} \dot{x}(t) = x^2 \\ x(0) = x_0 \end{cases} \quad (t \geq 0) \qquad \text{(also separable)}$$

Separate $\frac{dx}{dt} = x^2 \iff x^{-2}\,dx = dt$

$$-x^{-1} = t + C$$

$\Rightarrow$

$$X(t) = \frac{1}{c-t} \qquad \text{Impose: } x_0 \stackrel{!}{=} X(0) = \frac{1}{c}$$

So $$X(t) = \frac{1}{\frac{1}{x_0} - t} \qquad t \in [0, \tfrac{1}{x_0}).$$

The sol$^{n}$ exists... but only for a short time! OH NO!

OH MY! What to do?

---

Digression: FTC Part 1

Theorem: Suppose [f is continuous] on $x \in [a,b]$ then:

(1) for any $x_0 \in [a,b]$

$$\int_{x_0}^{x} f(t)\,dt := F(x) \quad \text{is differentiable}$$

and $F'(x) = f(x) \quad x \in [a,b]$.

i.e. f has an antiderivative.

MVT: If $F'(x) = f(x) = G'(x)$, then $F(x) = G(x) + C$