# PHD COMPREHENSIVE EXAM
## (Partial Differential Equations)
January, 2017

*Do any three of the four problems below. Show all steps and justify your answers. Each problem is worth ten points.*

1. Using the method of characteristics, solve for $u = u(x, y), (x, y) \in \mathbb{R}^2, x > 0$ the Cauchy problem

$$u_x + u_y = u^2 \text{ in the region } (x, y) \in \mathbb{R}^2, x > 0; \; u(x, -x) = x, (x > 0).$$

   Find an explicit formula for $u$ and show that the solution becomes infinite along the hyperbola $x^2 - y^2 = 4$.

2. Let $u$ solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty), \\ u = g, u_t = h \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

   Suppose $g$ and $h$ are twice continuously differentiable and have compact support. Let $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x, t)\, dx$ denote the *kinetic energy* and $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x, t)\, dx$ denote the *potential energy*.

   (a) Show that the total energy $E(t) := k(t) + p(t)$ is constant in $t$.

   (b) Show that $k(t) = p(t)$ for all large enough times $t$. (*Hint:* Assume $supp\ g$, $supp\ h \subset [a, b]$. Use d'Alembert's formula and show $u_t^2 - u_x^2 \equiv 0$ for all $x \in \mathbb{R}$ provided $t$ is sufficiently large.)

3. Let $U \subset \mathbb{R}$ be a bounded open interval in $\mathbb{R}$. Let $u : U \to \mathbb{R}$.

   (a) Define what it means to say that $u$ is weakly differentiable. Define the Banach space $W^{1,p}(U), 1 \le p \le \infty$.

   (b) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $F'$ being bounded. If $u \in W^{1,p}(U), 1 \le p < \infty$, show that $v = F(u)$ belongs also to $W^{1,p}(U)$ and that $v' = F'(u)u'$, where $u'$ and $v'$ denotes the corresponding weak derivatives while $F'$ denotes the classical derivative of $F$. Clearly state all the facts you are using to prove it.

4. Let $U \subset \mathbb{R}$ be an open and bounded subset of $\mathbb{R}^n$ and let

$$A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1}^n, \mathbf{b}(\mathbf{x}) = (b_1(\mathbf{x}), \cdots, b_n(\mathbf{x})), \mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), \cdots, c_n(\mathbf{x})),$$

   and $a(\mathbf{x})$ be respectively matrix-valued, vector-valued and scalar valued functions defined on $U$ with $\|A\|_{L^\infty(U)} < \infty$, $\|\mathbf{b}\|_{L^\infty(U)} < \infty$ and $\|\mathbf{c}\|_{L^\infty(U)} < \infty$. Assume moreover that there exists $\alpha > 0, M > 0$ such that

$$\alpha|\xi|^2 \le A(\mathbf{x})\xi \cdot \xi \le M|\xi|^2 \; \forall \; \xi \; \in \mathbb{R}^n \text{ and}$$

$$\frac{1}{2}\nabla \cdot (\mathbf{b} - \mathbf{c}) + a \geq 0 \ a.e. \text{ for } \mathbf{x} \in U.$$

Let $\mathcal{E}u = -\nabla \cdot (A(\mathbf{x})\nabla u - \mathbf{b}(\mathbf{x})u) + \mathbf{c}(\mathbf{x}) \cdot \nabla u + a(\mathbf{x})u$ and for $f \in L^2(U)$ consider the problem

$$\mathcal{E}u = f, u|_{\partial U} = 0. \qquad (*)$$

We say that $u$ is a weak solution of $(*)$ if $u \in H_0^1(U)$

$$\int_U \{A(\mathbf{x})\nabla u \cdot \nabla v - \mathbf{b}u \cdot \nabla v + \mathbf{c}v \cdot \nabla u + auv\} \, d\mathbf{x} = \int_U fv \, d\mathbf{x} \ \forall \ v \in H_0^1(U).$$

(a) State the Lax-Milgram theorem.

(b) Justifying all steps, use the Lax-Milgram theorem to show that a unique weak solution to $(*)$ exists which moreover satisfies the stability estimate

$$\|u\|_{H_0^1(U)} \leq C_\alpha \|f\|_{L^2(U)},$$

for an adequate constant $C_\alpha > 0$.