PhD COMPREHENSIVE EXAM IN

PARTIAL DIFFERENTIAL EQUATIONS

August 2012

Do any 3 of the 4 problems. Show all work. Each problem is worth ten points.

Q1. Describe the solutions $u(x,t)$ to Burger's equation

$$\begin{aligned} &u_t + uu_x = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ &u(x,0) = g(x), \quad x \in \mathbb{R}, \end{aligned}$$

for (a)

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and (b)

$$g(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Q2. On a bounded domain $\Omega \subset \mathbb{R}^n$, let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution to

$$\begin{aligned} &\Delta u - \mu^2 u = f \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\partial\Omega$ is the smooth boundary of Ω.

(a) Show $\max_{\Omega} |u| \leq \frac{1}{\mu^2} \max_{\Omega} |f|$.

(b) Use the result in part (a) to show that the smooth solutions of the problem

$$\begin{aligned} &\Delta u - \mu^2 u = f \quad \text{in } \Omega, \\ &u = g \quad \text{on } \partial\Omega, \end{aligned}$$

are unique.

Q3. For bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, consider the initial/boundary value problem for $u(x,t)$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u && \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) &= f(x) && x \in \Omega, \end{aligned}$$

where $\partial u/\partial \nu$ is the derivative of u in the outward normal direction at the boundary.

With $u(x,t)$ being a smooth solution,

(a) Assume $\displaystyle\int_\Omega f(x)\,dx = 0$. Show that $\displaystyle\int_\Omega u(x,t)\,dx = 0$ for all $t > 0$.

(b) Show that $\displaystyle\int_\Omega u(x,t)^2\,dx$ decays exponentially as $t \to \infty$.

Q4. Consider $u_{tt} - u_{xx} = 0$ on the quarter plane $x > 0$, $t > 0$, with $u(0,t) = 0$ for $t > 0$. Let $u(x,0) = \max\{0, (x-1)(3-x)\}$, which has its peak at $x = 2$.

(a) By following the characteristics of the wave equation, at what location(s) will the 'initial data at $x = 2$' find itself when $t = 3$?

(b) What is the *domain of dependence* of the point $(x,t) = (1,10)$?

(c) What is the *region of influence* of the point $(x,t) = (10,0)$?