3. <u>Well-Posedness</u> (3 pts) What three conditions are required for a problem to be well-posed?

1. Short answer questions (a)-(e):

   (a) Specify the type of equation (elliptic, hyperbolic, or parabolic) for the equation $u_y = u_{xx} - 2u_{xy} + 3u_{yy}$ in the plane.

   (b) What is the general solution to a 1D wave equation, and what is its meaning?

   (c) Name two properties that distinguish differences of behavior of the solutions to the Cauchy problem for the wave and heat problems.

   (d) What is the *domain of dependence* for the pint $(x, t) = (4, 6)$, given the equation $u_{tt} = 4u_{xx}$?

   (e) State at least two of the three conditions for a problem to be well-posed.

1. Short answer questions (a)-(g):

   (a) Mark which of the following equations are linear, and which are nonlinear:

      i. $\frac{\partial^2 u}{\partial x \partial y} + u = 0$

      ii. $\tan(x/y)\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial u}{\partial y} - \sqrt{1+x^2}\cos(y)u = e^{-x}$

      iii. $\frac{\partial u}{\partial t} + e^{-u}\frac{\partial u}{\partial x} = 1$

   (b) Specify the type of equation (elliptic, hyperbolic, or parabolic) given by $u_{xx} - 3u_{xy} + 2u_{yy} + (u_x)^2 + u^2 = 0$ in the plane. Explain your answer.

   (c) If a point stimulus arises at location $x = 5$ at time $t = 0$ and is governed by $u_{tt} = \frac{1}{100}u_{xx}$, where will the signal be at time $t = 20$?

   (d) For the equation $u_{tt} = \frac{1}{36}u_{xx}$, what is the *region of influence* of the point $(x, t) = (1, 0)$?

   (e) Is Burger's equation $u_t + uu_x = 0$ linear, semilinear, quasilinear, or none of these?

   (f) The general solution to the heat equation $u_t = Du_{xx}$ is $u(x,t) = C_1 \int_0^{x/\sqrt{4Dt}} e^{-s^2}\, ds + C_2$. What is the solution to the Cauchy problem when $u(x, 0) = 1 - 2H(-x)$, where $H(\cdot)$ is the Heaviside function. (Recall, $\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}/2$.)

   (g) Name two properties that distinguish differences of behavior of the solutions to the Cauchy problem for the wave and heat problems.

   (h) State at least two of the three conditions for a problem to be well-posed.

1. Short answer questions, do not calculate more than necessary

(a) Give a qualitative property of a solution to the *Cauchy problem* for the heat equation that the solution of the wave equation does **not** have.

(b) State the Mean Value Theorem for a harmonic function u, defined on some general domain $\Omega \subset \mathbb{R}^2$.

(c) For the eigenvalue equation $\frac{d^2\phi}{dx^2} + \lambda e^x \phi = 0$, $0 < x < 5$, with $\frac{d\phi}{dx}(0) = 0 = \phi(5)$ is a regular Sturm-Liouville eigenvalue problem. What can be said of the eigenfunctions for two different eigenvalues?

(d) For the Cauchy problem $u_{tt} = \frac{1}{4}u_{xx}$, what is the *domain of dependence* for the point $(x, y) = (\cancel{11,} 12)$?

(e) If $f(x) = x(\pi - x)$ for $x \in [0, \pi]$, sketch the graph of the function the Fourier sine series converges to *uniformly* in $\mathbb{R}$.

(f) In the one-dimensional case, what kind of function is u if it is harmonic and bounded in all of $\mathbb{R}$?

(g) For the general Cauchy heat equation problem $u_t = Du_{xx}$, $u(x, 0) = e^{-x^2}$, $|x| < \infty$, $t > 0$, write out the solution in terms of the fundamental solution of the heat equation.

(h) Consider the first-order problem $xyu_x + yu_y = u^2$, $u(x, -x) = \sin(x)$, $x > 0$. Write out the characteristic system (three equations plus their initial conditions) for solving the first-order Cauchy problem, but **do not** solve them.

(i) For the Bernoulli-Euler beam equation problem, what would be the boundary condition at $x = 0$ if it is specified that the beam is *free* at that end?

(j) Let $u_{xy} + u = 0$ in $\Omega = \{(x, y) : x > 0, y > 0\}$. What type equation (elliptic, parabolic, hyperbolic) equation is it in Ω?

(a) Specify the type of equation (elliptic, hyperbolic, or parabolic). Explain your answer.

$$2\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial y \partial x} + 6\frac{\partial^2 u}{\partial y^2} - 2.5\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = e^{-x} \quad \text{in the plane}$$

(b) For the Cauchy problem $u_{tt} = 0.09u_{xx}$, $|x| < \infty$, $t > 0$,what is the *domain of dependence* for the point $(x, t) = (-3, 10)$?

(c) For the same equation as in part (b), what is the *region of influence* of the point $(x, t) = (-3, 0)$?

(d) Name two properties that distinguish differences of behavior of the solutions to the Cauchy problem for the wave and heat problems.

(e) Indicate which of the following equations are linear, and which are nonlinear:

i. $y^2 u_x + x^2 u_y = 1/u$

ii. $u_{xx} + uu_y = \cos(x)$

iii. $\sqrt{1 + x^2}\cos(y)u_x + 3\sin(x)u_{xy} - u = \cos(2x)$

1. *Short answer questions; do not calculate more than necessary*

(a) For the eigenvalue equation $\frac{d}{dx}\{p(x)\frac{d\phi}{dx}\} + \lambda\phi = 0,\ 0 < x < 10$, what conditions on $p(x)$ make this a regular Sturm-Liouville eigenvalue equation?

(b) State the Strong Maximum Principle for harmonic function u, defined on some general domain $\Omega \subset \mathbb{R}^2$.

(c) Extend the function $f(x) = x^2$ for $x \in [0, 1]$ to $[-1, 1]$ such that its Fourier series converges *uniformly* to the extended function on $[-1, 1]$.

(d) Let $u(x, t)$ be a solution to $u_{tt} - 0.25u_{xx} = 0$ in $\Omega = \{(x, t) : |x| < \infty, t > 0\}$. If we follow an disturbance that initiates at $x = 2$ at time $t = 0$ along the characteristics, what location(s) will this disturbance be at $t = 6$?

(e) For the equation $u_{tt} - 4u_{xx} = 0$ in $\Omega = \{(x, t) : |x| < \infty, t > 0\}$, what is the *region of influence* of the point $(x, t) = (13, 0)$?

(f) For the general Cauchy heat equation problem $u_t = u_{xx}$, $u(x, 0) = e^{-2x^2}$, $|x| < \infty$, $t > 0$, write out the solution in terms of the fundamental solution of the heat equation.

(g) Give a qualitative property of a solution to the Cauchy problem for the heat equation that the solution of the wave equation does not have.

(h) Consider the first-order problem $(x + y)u_x + 3yu_y = u$, $u(x, -x) = \sin(x)$, $x > 0$. Write out the parameterized description of the initial curve Γ and then the characteristic system (three equations plus their initial conditions) for solving the first-order Cauchy problem, but **do not** solve them.

1. (a) For the eigenvalue equation $\frac{d}{dx}(p(x)\frac{d\varphi}{dx}) + \lambda\varphi = 0, \quad 0 < x < 1$, what are the conditions on *p(x)* to make this a regular Sturm-Liouville equation?

(b) The solution to the 1D heat equation on the line

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad |x| < \infty, \quad t > 0, \quad u(x,0) = \sin(x),$$

is written in terms of a heat kernel $S(x,t)$, i.e. $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\sin(y)dy$.

What is the specific heat kernel $S(x,t)$?

(c) For harmonic functions on Ω, that is, solutions to $\nabla^2 u = 0$ on Ω, what is the maximum principle?

(d) Given the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad |x| < \infty, \quad t > 0,$$

what is the domain of dependence for the point $(x,t) = (4,3)$? (Hint: draw the characteristics through (4,3).)

1. *Short answer questions; do not calculate more than necessary*

   (a) For the eigenvalue equation $\frac{d}{dx}\{p(x)\frac{d\phi}{dx}\} + \lambda\phi = 0$, $0 < x < 5$, what conditions on $p(x)$ make this a regular Sturm-Liouville eigenvalue equation?

   (b) Let $f(x) = \sin(x)$ on $(-\pi/2, \pi/2)$. The Fourier series for f converges for every $x \in [-\pi/2, \pi/2]$. Sketch the graph of the function the series $\frac{a_0}{2} + \sum_{n\geq 1}\{a_n \cos(2nx) + b_n \sin(2nx)\}$ represents on $[0, 3\pi/2]$.

   (c) Consider the Cauchy problem $xu_x - x^2 u_y + 3u = 0$, $u(x, x) = 3x^3$. Write out the characteristic system (three equations plus their initial conditions) for solving the first-order Cauchy problem, but **do not** try to solve them.

   (d) If $u_{tt} - 4u_{xx} = 0$ is given on the quarter plane, i.e. for $x > 0$, $t > 0$, with $u(0, t) = 0$ for $t > 0$. Assume a single jump discontinuity in the initial displacement (i.e., at time $t = 0$) at location $x = 2$. By following the characteristics, what location(s) will the singularity be at when $t = 5$?

   (e) If $u(r, \theta)$ is a harmonic function on the unit disk $\Omega = \{(r, \theta) : 0 \leq r < 1\ ,\ 0 \leq \theta < 2\pi\}$, what is $u(0, 0)$, given the boundary values

$$u(1, \theta) = f(\theta) = \begin{cases} \pi & \text{if } 0 \leq \theta < \pi \\ -2 & \text{if } \pi \leq \theta < 3\pi/2 \\ 6 & \text{if } 3\pi/2 \leq \theta < 2\pi \end{cases}$$

   (f) For the wave equation $u_{tt} - 9u_{xx} = 0$ for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, what is the domain of dependence for the point $(x, t) = (8, 9)$?

   (g) In the Cauchy heat problem $u_t = Du_{xx} + F(x, t)$, $|x| < \infty$ $\quad t > 0$, $u(x, 0) = 0$, the solution can be written in the form $u(x, t) = \int_0^t \int_{-\infty}^{\infty} S(x - y, t - \tau)F(y, \tau)dy\, d\tau$, where $S(x, t)$ is the fundamental solution of the heat equation. What explicitly is $S(x, t)$?

   (h) State the maximum principle for harmonic functions, u, defined on domain $\Omega \subset \mathbb{R}^2$.

1. Short answer quiz

(a) Let $f(x) = \begin{cases} \cos(x) & 0 \le x < \pi/2 \\ -\cos(x) & -\pi/2 \le x < 0 \end{cases}$. The Fourier series for f converges for every $x \in [-\pi/2, \pi/2]$. Sketch the graph of the function the series $\frac{a_0}{2} + \sum_{n\ge1}\{a_n \cos(2nx) + b_n \sin(2nx)\}$ represents on $[-\pi, \pi]$.

(b) For the eigenvalue equation $\frac{d}{dx}\{p(x)\frac{d\phi}{dx}\} + \lambda\phi = 0,\ 0 < x < 10$, what conditions on $p(x)$ make this a regular Sturm-Liouville eigenvalue equation?

(c) If $u_{tt} - u_{xx} = 0$ for $x > 0,\ t > 0$, with $u(0,t) = 0$ for $t > 0$, assume a single point disturbance is initiated at time $t = 0$ at location $x = 1$. By following the characteristics, what location(s) will the disturbance be at when $t = 2$?

(d) If $u(r,\theta)$ is a harmonic function on the unit disk $\Omega = \{(r,\theta) : 0 \le r < 1\ ,\ 0 \le \theta < 2\pi\}$, what is $u(0,0)$, given the boundary values

$$u(1,\theta) = f(\theta) = \begin{cases} 12 & \text{if } 0 \le \theta < \pi/4 \\ -4 & \text{if } \pi/4 \le \theta < \pi \\ \pi & \text{if } \pi \le \theta < 2\pi \end{cases}$$

(e) In the Cauchy heat problem $u_t = Du_{xx},\ |x| < \infty\quad t > 0,\ u(x,0) = f(x)$, the solution can be written in the form $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)f(y)dy$, where $S(x,t)$ is the fundamental solution of the heat equation. What explicitly is $S(x,t)$?

(f) State the maximum principle for harmonic functions, u, defined on domain Ω.

(g) Consider the Cauchy problem $3y\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial y} + 5u = 0,\ u(x,x) = F(x)$. Write out the characteristic system (three equations plus initial conditions) for solving the first-order Cauchy problem, but **do not** go further and solve the system.

1. Short answer problem:

a) For the Cauchy problem $\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2}$, $|x| < \infty$, t>0, $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$,

what is d'Alembert's formula for the solution?

b) If $f(x) = e^{-x^2}$ ($g(x) = 0$) for the problem in a), the initial displacement is a single "bump". Since this bump travels as a wave, where is the maximum of $u(x,t)$ when $t = 10$?

c) In what part of the plane is the following operator elliptic?

$$Lu = \frac{\partial}{\partial x}\left(e^{-x}\frac{\partial u}{\partial x}\right) - 2x(x+1)\frac{\partial^2 u}{\partial y^2} + e^{-x-y}\frac{\partial u}{\partial x}$$

d) If $\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + \lambda\sigma(x)\varphi = 0$, on $a < x < b$, is a regular Sturm-Liouville eigenvalue equation, then what conditions are imposed on $\sigma(x)$?

e) If $u(r,\theta)$ is a harmonic function on the unit disk $\Omega = \{(r,\theta) \mid 0 \le r < 1, 0 \le \theta < 2\pi\}$,

and has boundary values $u(1,\theta) = \begin{cases} 90 & -\pi/2 < \theta < \pi/2 \\ 25 & \pi/2 < \theta < \pi \\ 7 & \pi < \theta < 3\pi/2 \end{cases}$, what is $u(0,0)$?

f) For the function described in part e), what bounds are on $u(r,\theta)$ for (r,θ) in the unit disk (i.e. what are A, B if $A \le u(r,\theta) \le B$)?

Problem 1 (5pts) *Verify that* $u(x,y) = \ln\sqrt{x^2+y^2}$ *satisfies the Laplace equation*

$$u_{xx} + u_{yy} = 0$$

for all $(x, y) \neq (0,0)$.
solution:

1. **2nd Order Classification** (8 pts) Classify the following problem where appropriate by (1) Order, (2) Linear of Nonlinear, (3) Homogeneous or Non-homogeneous, and (4) Elliptic, Hyperbolic, or Parabolic.

$$u_t - (xu_x)_x = f(x,y), \quad \{(x,t) : |x| < \infty, t > 0\}$$

4. A model for a population with density $u(x,t)$ in an advection-driven environment, like a stream, might be given by

$$u_t + \nu u_x = Du_{xx} + ru \qquad |x| < \infty\ ,\ t > 0\ .$$

Let $u(x,0) =$ constant $= k > 0$, $|x| < \infty$. Let $u(x,t) = e^{ax-bt}w(x,t)$.

(a) Reduce the u equation to a diffusion equation for w, i.e. $w_t = Dw_{xx}$, by finding the proper values for a, b.

(b) Write the correct initial condition for w.

(c) Write out the formula for $w(x,t)$, but do **not** bother to integrate the expression.

1. **In the plane determine the type of equation (elliptic, hyperbolic, or parabolic). Briefly explain your answer. If the equation changes type in the domain, explain where it is which type.**

(a) $\dfrac{\partial^2 u}{\partial x^2} - 6\dfrac{\partial^2 u}{\partial y \partial x} + 12\dfrac{\partial^2 u}{\partial y^2} = 0$ in the plane

(b) $\dfrac{\partial^2 u}{\partial x^2} + 2y\dfrac{\partial^2 u}{\partial x \partial y} + x\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial x} + u = 0$ for $x, y > 0$

Problem 1 (5pts) *Verify that* $u(x,y) = f(x)g(y)$ *is a solution of the PDE*

$$uu_{xy} = u_x u_y$$

for all pairs of (differentiable) functions f *and* g *of one variable.*
solution:

2. What is the general solution of $3\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 u}{\partial x \partial y} = 0$ in terms of two general functions?
(Hint: think of the equation as an ODE in $\partial u / \partial y$)

2. For what values of non-zero constants a and b is $w(x,t) = e^{ax-bt}\cos(x)$ a solution to

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial w}{\partial x} \quad ?$$

1. **1st Order Transport Equation** Consider the following equation

$$u_t + cu_x = \frac{1}{x}, u(x,0) = e^{-x^2}, |x| < \infty, t > 0, \quad c = \text{constant}$$

(a) (3 pts) Construct the characteristic equations including initial conditions

(b) (3 pts) Solve the system.

Problem 2 *(20pts) Use characteristic coordinates to solve the initial value problem $u_t + u_x = -tu$, $x \in \mathbb{R}$, $t > 0$; $u(x,0) = f(x)$, $x \in \mathbb{R}$.*
solution:

5. **Heat Equation**

(a) (3 pts) Write the heat source function, $S(x,t)$, (i.e. heat kernel).

(b) (6 pts) Write the solution to the equation $u_t = Du_{xx} + e^{-t}\delta(x)$ on $|x| < \infty, t > 0$, $u(x,0) = 0$ for $|x| < \infty$. You may use Duhamel's principle or not. You need not simplify.

2. Find the solution to the first-order Cauchy problem $2u_x - xu_y = x^2 + y$, $u(0,y) = 1$, $|y| < \infty$.

3. Obtain the solution to the first-order Cauchy problem $(x+2)u_x + 2yu_y = 2u$, $u(0,y) = y$, for $y \in \mathbb{R}$.

3. Solve the Cauchy problem $2\dfrac{\partial u}{\partial x} + 4x\dfrac{\partial u}{\partial y} = y \;,\; u(0,y) = y$.

2. Solve the Cauchy problem $\dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0 \;,\; u(0,y) = 1 + y^2$.

3. Consider the wave equation problem

$$u_{tt} = \tfrac{1}{64}u_{xx} \qquad |x| < \infty \;,\; t > 0$$

$$u(x,0) = xH(x) = \begin{cases} x & x > 0 \\ 0 & x \le 0 \end{cases}, \qquad u_t(x,0) = 0.25H(x) = \begin{cases} 1/4 & x > 0 \\ 0 & x \le 0 \end{cases}$$

(a) Break the domain into regions using the appropriate characteristics ($c = 1/8$), and determine the solution in each of the regions.

(b) Sketch the graph of $u(x,8)$.

Problem 2 *(15pts) Find the general solution of $u_t + 2u_x - u = t$. (Hint: use characteristic coordinates.)*
solution:

2. **1st Order Transport Equation** Consider the following equation

$$u_t + v(t)u_x = -u, u(x,0) = e^{-x^2}, |x| < \infty, t > 0$$

(a) (6 pts) Construct the characteristic equations including initial conditions

(b) (6 pts) Solve the system.

5. **Heat Equation** (4 pts) Write the solution to the equation $u_t = Du_{xx} + e^{-(x-x_0)^2}$ on $|x| < \infty, t > 0$, $u(x,0) = 0$ for $|x| < \infty$.

2. Obtain the solution to the first-order Cauchy problem $2xu_x - 2yu_y = u$, $u(x,1) = x^2$.

3. Cauchy Problem - Wave Equation

$$u_{tt} - c^2 u_{xx} = 0, \quad |x| < \infty, t > 0, \quad u(x,0) = f(x), u_t(x,0) = g(x), \quad |x| < \infty$$

(a) (6 pts) Show that $u(x,t) = F(x - ct) + G(x + ct)$ is a solution to the wave equation above.

(b) (6 pts) Assuming the initial data is smooth write d'Alembert's solution for the given problem.

(c) (2 pts) Define the Domain of Dependence for the point $(x,t) = (3,2)$, if $c = 2$.

5. Solve the Cauchy problem $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = -1$, with $u(x,0) = \sin(x)$, $|x| < \infty$.

2. Write the solution to the following wave equation IVP.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad |x| < \infty \quad t > 0$$

$$u(x,0) = 5 + 4H(x-1) = \begin{cases} 9 & x > 1 \\ 5 & x \leq 1 \end{cases}, \quad u_t(x,0) = \cos(x/2)$$

(Look at the 3 regions of the domain broken up by the characteristics $x \pm t = 1$ since $c = 1$.)

Problem 4 ***(10pts)*** *Let c_0 be constant. (a) Verify that $u = F(x - c_0 t)$ and $u = G(x + c_0 t)$ are solutions of the wave equation with wave speed c_0 for any twice-differentiable functions F and G, (b) sketch the region of influence and the domain of dependence.*

solution:

2. **Cauchy Problem - Wave Equation** (6 pts) Solve the following problem

$$u_{tt} - 4u_{xx} = 0, \quad |x| < \infty, \quad t > 0$$

$$u(x,0) = \sin(x), u_t(x,0) = \cos(x)$$

5. Write the solution to the following wave equation IVP. (Look at the 3 regions of the plane broken up by the characteristics $x \pm t = 1$.)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad |x| < \infty, \quad t > 0$$

$$u(x,0) = \quad \frac{\partial u}{\partial t}(x,0) = 3 + 4H(x-1) = \begin{cases} 7 & x \geq 1 \\ 3 & x < 1 \end{cases}$$

4. (a) Write out d'Alembert's formula for the solution to the 1D wave equation problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & |x| < \infty, t > 0 \\ u(x,0) = f(x) & |x| < \infty \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \end{cases}$$

Then use the formula to write the solution when $c = 1$, $f(x) = \exp(-x^2)$, *and* $g(x) = 0$.

(b) Write the formula for the solution to the 1D heat equation problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} & |x| < \infty, t > 0 \\ u(x,0) = f(x) & |x| < \infty \end{cases}$$

Then use the formula to determine the solution when $D = 1$, $u(x,0) \equiv 1$

3. What is the solution to the 1D wave equation problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2} & |x| < \infty, t > 0 \\ u(x,0) = 0 & |x| < \infty \\ \dfrac{\partial u}{\partial t}(x,0) = xe^{-x^2} & |x| < \infty \end{cases}$$

3. Write the solution to the following wave equation IVP.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad |x| < \infty \quad t > 0$$

$$u(x,0) = 1 + xH(x-1) = \begin{cases} 1 & x \le 1 \\ 1+x & x > 1 \end{cases}, \quad u(x,0) = 0$$

(Look at the 3 regions of the domain broken up by the characteristics $x \pm t = 1$ since $c = 1$.)

Problem 3 *(20pts) Use Laplace transform to solve* $u_t = u_{xx}$ *on* $x, t > 0$ *with* $u(0,t) = a$, $t > 0$ *and* $u(x,0) = b$, *where* a *and* b *are constants.*
solution:

6. **Integral Transforms** Transform the diffusion equation $u_t = u_{xx}$ on $|x| < \infty, t > 0$, $u(x,0) = f(x)$ for $|x| < \infty$.

(a) (3 pts) Using Laplace transform, set-up the equation in Laplace space, but DO NOT solve or transform back.

(b) (3 pts) Using Fourier transform, set-up the equation in Fourier space, but DO NOT solve or transform back.

3. Consider the problem

$$\frac{\partial^2 v}{\partial t^2} = 9\frac{\partial^2 v}{\partial x^2} \quad x > 0, t > 0$$

$$v(x,0) = A, \quad \frac{\partial v}{\partial t}(x,0) = 0$$

$v(0,t) = B$, $v(x,t)$ remains bounded as $x \to \infty$, A and B are positive constants

a) Use the Laplace transform to convert the problem to an ODE problem
b) Solve the problem in part a)
c) Invert the problem using the Laplace transform table to obtain $v(x,t)$.

7. Apply the Laplace transform $U(x,s) = \int_0^\infty e^{-st}u(x,t)dt$ to the following problem to obtain the formula for $U(x,s)$. Do not invert.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad |x| < \infty, \quad t > 0, \; u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \cos x, \text{ u remains bounded as}$$

$|x| \to \infty$, for all $t > 0$. (Recall, for appropriate f, the Laplace transform of $\dfrac{d^n f}{dt^n}$ is

$$s^n F(s) - \sum_{j=1}^{n} s^{n-j} \frac{d^{j-1} f}{dt^{j-1}}(0).)$$

6. **Integral Transforms** Transform the wave equation $u_{tt} = u_{xx}$ on $|x| < \infty, t > 0$, $u(x,0) = f(x)$ for $|x| < \infty$, $u_t(x,0) = g(x)$ for $|x| < \infty$.

(a) (3 pts) Using Laplace transform and initial conditions, set-up the equation in Laplace space, but DO NOT solve or transform back.

(b) (3 pts) Using Fourier transform, set-up the equation and initial conditions in Fourier space, but DO NOT solve or transform back.

3. Consider the problem

$$f(x) = \begin{cases} u_{tt} = u_{xx} + 4e^{-2x} & 0 < x < 1\ ,\ t > 0 \\ u_x(0,t) = 2\ ,\ u(1,t) = 0 & t > 0 \\ u(x,0) = e^{-2x}\ ,\ u_t(x,0) = 0 & 0 < x < 1 \end{cases}$$

(a) Let $u(x,t) = U(x) + w(x,t)$, where $U(x)$ is the steady state solution. Determine $U(x)$.

(b) Given $U(x)$, determine the problem $w(x,t)$ is to solve. (But do **not** solve for w.)

5. Consider the forced heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-x} & 0 < x < 1\ ,\ t > 0 \\ u(0,t) = 0\ ,\ u_x(1,t) = 1 & t > 0 \\ u(x,0) = x + 1\ , & 0 < x < 1 \end{cases}$$

Let the solution be written in the form $u(x,t) = U(x) + w(x,t)$, where $U(x)$ is the steady state solution. Determine $U(x)$.

4. Consider the heat equation problem

$$\begin{cases} u_t = u_{xx} + 4u + 1 + \sin(x) & 0 < x < 1\ ,\ t > 0 \\ u(0,t) = 0 = u(1,t)\ ,\ t > 0 & u(x,0) = 0\ ,\ 0 < x < 1 \end{cases}$$

Determine the steady state solution $u = U(x)$.

3. Consider the forced heat equation problem $u_t = u_{xx} + x^5$ on $0 < x < 1$, $t > 0$, with $u_x(0,t) = \frac{1}{42}, u(1,t) = 0$.

(a) Find the **steady state solution** $U(x)$.

(b) If we write $u(x,t) = U(x) + w(x,t)$ for the solution of the non-homogeneous problem, then from the PDE and boundary conditions we find that $w(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n-1/2)^2\pi^2 t} \cos[(n-1/2)\pi x]$. For large t about where will the hottest spot be along the rod with the heat distribution modeled by this model?

3. Consider the wave equation problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 25e^{-x} & 0 < x < 1\ ,\ t > 0 \\ \frac{\partial u}{\partial x}(0,t) = 1\ ,\ u(1,t) = 0 & t > 0 \\ u(x,0) = e^{-2x}\ ,\ \frac{\partial u}{\partial t}(x,0) = 0 & 0 < x < 1 \end{cases}$$

(a) Let $u(x,t) = U(x) + w(x,t)$, where $U(x)$ is the steady state solution. Determine $U(x)$.

(b) Given $U(x)$, what problem does $w(x,t)$ solve? (Do not solve it.)

3. Consider the heat equation problem (where $c > 0$ is a constant)

$$\begin{cases} u_t = u_{xx} - cu + \sin(\pi x) + \frac{1}{8}\sin(3\pi x) & 0 < x < 1 \,,\ t > 0 \\ u(0,t) = 0 = u(1,t) \,,\ t > 0 & u(x,0) = 0 \,,\ 0 < x < 1 \end{cases}$$

(a) Determine the steady state solution $u = U(x)$.

(b) If, in the full problem, solution u is written in the form $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$, set up the differential equations and associated initial conditions for $a_n(t)$, $n = 1, 2, 3, \ldots$. You need **not** have to solve the ODEs.

2. Consider the problem
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + e^{-x}, & 0 < x < 1, t > 0 \\ \dfrac{\partial u}{\partial x}(0,t) = 1, u(1,t) = 2 & t > 0 \\ u(x,0) = 2x & 0 < x < 1 \end{cases}$$

a) Let $u(x,t) = U(x) + w(x,t)$, where $U(x)$ is the steady state solution. Determine $U(x)$.

b) Given $U(x)$, what problem does $w(x,t)$ solve? (Do not solve it!)

2. Given the problem
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < \pi, & t > 0 \\ u(x,0) = 0 & & 0 < x < \pi \\ \dfrac{\partial u}{\partial x}(0,t) = 0, & u(\pi,t) = t, & t > 0 \end{cases}$$

Transform the problem to a problem with homogeneous boundary conditions. (Do <u>not</u> solve the problem.)

4. Consider the wave equation problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 2.5e^{-x/10} & 0 < x < 1 \,,\ t > 0 \\ u(0,t) = 10 \,,\ \frac{\partial u}{\partial x}(1,t) = 0 & t > 0 \\ u(x,0) = 0 \,,\ \frac{\partial u}{\partial t}(x,0) = 0 & 0 < x < 1 \end{cases}$$

(a) Let $u(x,t) = U(x) + w(x,t)$, where $U(x)$ is the steady state solution. Determine $U(x)$.

(b) Given $U(x)$, what problem does $w(x,t)$ solve? (Do **not** solve it.)

3. Consider $f(x) = \pi - x$ on $0 < x < \pi$. Sketch the graph of the function that the Fourier sine series $\sum_1^\infty b_n \sin(nx)$ for $f(x)$ converges to (pointwise) on the real line. (You only need to sketch the graph over, say, the interval $-2\pi \le x \le 2\pi$.) Compute the coefficients of this Fourier sine series. (Useful fact: $\cos(n\pi) = (-1)^n$)

8. **Fourier Series** (4 pts) If $f(x) = e^{-x}$ for $x \in (0,1)$, extend the function (oddly or evenly) to [-1,1] so that the Fourier series for the extension of $f(x)$ converges uniformly in $(-\infty, \infty)$.

1. Let
$$f(x) = \begin{cases} 1 & -1 < x \le 0 \\ 1 - x & 0 < x < 1 \end{cases}$$

(a) The Fourier series for $f(x)$ converges pointwise **for every** $x \in \mathbb{R}$. Sketch the graph the Fourier series converges to on $[-3, 3]$.

(b) To see that you can determine Fourier coefficients, computing just one coefficient will be enough. With $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos(n\pi x) + b_n \sin(n\pi x))$,

2. The function f, which is to be *continuous* in $(-1, 1)$, is given by
$$f(x) = \begin{cases} mx + b & \text{for } -1 < x \le 0 \\ 2x - 1 & \text{for } 0 < x < 1 \end{cases}$$

(a) For what value(s) of m and b (if any) will the full Fourier series of $f(x)$ converge **pointwise** on $[-1, 1]$?

(b) For what value(s) of m and b (if any) will the full Fourier series of $f(x)$ converge uniformly on $[-1, 1]$?
(Draw yourself a picture and recall the hypotheses of the convergence theorems)

3. If $f^*(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}$, where
$$f^*(x) = \begin{cases} 2x & \text{for } -1 < x \le 0 \\ 0 & \text{for } 0 < x \le 1, \end{cases}$$
compute a_0 and b_1 (only).

2. Consider the function
$$f(x) = \begin{cases} mx + b & \text{for } -1 < x \le 0 \\ 2x - 1 & \text{for } 0 < x < 1 \end{cases}$$

(a) For what value(s) of m and b (if any) will the full Fourier series of $f(x)$ converge **pointwise** on $(-1, 1)$?

(b) For what value(s) of m and b (if any) will the full Fourier series of $f(x)$ converge uniformly on $(-1, 1)$?
(draw yourself a picture and recall the hypotheses of the convergence theorems)

(c) If $f^*(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}$, where
$$f^*(x) = \begin{cases} 2x & \text{for } -1 < x \le 0 \\ 0 & \text{for } 0 < x \le 1, \end{cases}$$
compute a_0 and b_1 (only).

8. **Fourier Series** (5 pts) If $f(x) = 1 - x$ for $x \in (0, 1)$, extend the function to $[-1,1]$ so that the Fourier series of $f(x)$ converges *pointwise* but not uniformly in $(-\infty, \infty)$. Explain why. Sketch this Fourier series representation of $f(x)$ for a large number of terms in the partial sum on (-3,3).

2. Consider the function

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 2/3 \\ 0 & \text{for } 2/3 \le x \le 1 \end{cases}$$

Consider the sine series for $f(x)$ on $[0, 1]$, that is $f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x)$. Note that the function is piecewise smooth on the interval $[0, 1]$.

(a) Sketch a graph of the function that is defined on $\mathbb{R}$ that the series converges to. Restrict your graph to $[-1, 3]$.

(b) Compute one Fourier coefficient, say b_1.

4. a) Define $f(x) = x - \dfrac{1}{2}x^2$ on interval $0 < x < 1$. If *f(x)* was expanded in a Fourier series of eigenfunctions determined from the eigenvalue problem $\dfrac{d^2\varphi}{dx^2} + \lambda\varphi = 0, \;\; 0 < x < 1,$

$\varphi(0) = 0 = \dfrac{d\varphi}{dx}(1)$, $f(x) \sim \sum_{n=1}^{\infty} a_n \varphi_n(x)$, write the formula for a_n, but **do not compute it.**

b) Sketch the graph of the function on the interval [0,3] that the Fourier series converges to pointwise, considered defined on the real line.

4. Consider $f(x) = 2x + \dfrac{1}{2}$ on $0 < x < \pi$. (a) Sketch the graph of the function that the Fourier sine series $\sum_1^{\infty} b_n \sin(nx)$ for $f(x)$ converges to (pointwise) on the line. (You only need to sketch the graph over, say, the interval $-2\pi \le x \le 2\pi$.) (b) Compute the coefficients of this Fourier sine series. (Useful fact: $\cos(n\pi) = (-1)^n$).

Problem 6 ***(10pts)*** *Define the norm and inner product on space $L^2[a, b]$. If $f(x) \in L^2[a, b]$ and $\{f_n\}$ is an orthogonal system of L^2 functions on $[a, b]$, write $f(x)$ in a form of a generalized Fourier series, and define the Fourier coefficients. How would the series and the coefficients change if the system was orthonormal?*

solution:

4. Consider the problem $3\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x, y < 1$, with

$$u(0, y) = u(x,0) = u(x,1) = 0, \quad u(1, y) = \sin(2\pi \cdot y)$$

(a) Applying separation of variables, what is the eigenvalue problem, the resulting eigenvalues, and associated eigenfunctions?

(b) Write the eigenfunction expansion for *u(x,t)*, and then solve for the coefficients. Then write out the final form of *u(x,t)*.

1. Find the eigenvalues (only) to the problem

$$\frac{d^2\varphi}{dx^2} + \lambda\varphi = 0, \quad 0 < x < 1$$

$\varphi(0) = \varphi(1), \qquad \dfrac{d\varphi}{dx}(1) = 0$ (Assume, a priori, that $\lambda \ge 0$)

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad 0 < x < 1 \\ \phi(0) = \phi(1) \\ \frac{d\phi}{dx}(0) = 0 \end{cases}$$

It can be shown that the problem has no negative eigenvalues.

(a) Determine whether $\lambda = 0$ is an eigenvalue or not.

(b) Determine the positive eigenvalues of the problem. (Be careful about what values the particular trig function(s) take on; if need be, sketch for yourself the graph of the appropriate trig function.)

(c) Finally, give the associated eigenfunctions.

3. Given the problem

$$(1+x)^2\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{1+x}\frac{\partial u}{\partial x}\right) \text{ on } 0 < x < 1, \quad t > 0, \text{ with } u(0,t) = 0 = \frac{\partial u}{\partial x}(1,t), \quad t > 0,$$

separate variables and write out the eigenvalue problem (equation and boundary conditions). Do not solve for the eigenvalues or eigenfunctions.

Problem 7 *(10pts) If c_n are Fourier coefficients of f and f_n is an orthonormal set, show that*

$$\left(\sum_{n=1}^{N} c_n f_n, f - \sum_{n=1}^{N} c_n f_n\right) = 0.$$

solution:

Problem 8 *(15pts) Use the separation of variables to solve*

$$\begin{aligned} u_t &= k u_{xx}, \quad x \in (0,l), \quad t > 0, \\ u_x(0,t) &= 0, \quad u_x(l,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad x \in (0,l). \end{aligned}$$

(Set up the Sturm-Liouville problem correctly!)

1. Consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0 = \frac{\partial u}{\partial x}(\pi, t) & t > 0 \end{cases}$$

(a) Separate variables, and derive the EVP and the time-dependent equation.

(b) Solve for the eigenvalues and associated eigenfunctions. (All eigenvalues are positive in this problem, so you need not check other cases.)

Problem 8 *(15pts) Use the separation of variables to solve*

$$\begin{aligned} u_t &= k u_{xx}, \quad x \in (0,l), \quad t > 0, \\ u(0,t) &= 0, \quad u(l,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad x \in (0,l). \end{aligned}$$

(Set up the Sturm-Liouville problem correctly!)
solution:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & 0 < x < 2\pi \,,\ t > 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(2\pi,t) & t > 0 \\ u(x,0) = 7\cos(10x) & 0 < x < 2\pi \end{cases}$$

When we separate variables and solve the eigenvalue problem, we find $\lambda = 0$ is an eigenvalue (with constant associated eigenfunction), and the other eigenvalues are positive and given by $\lambda_n = n^2/4$, with the associated eigenfunctions being $\phi_n(x) = \cos(\frac{nx}{2})$. This means, with $u(x.t) = T(t)\phi(x)$, that $T(t) = T_n(t) = e^{-n^2t/4}$.

(a) Write out the series for $u(x,t)$, then let $t = 0$ to get the cosine series for $u(x,0)$.

(b) Substitute in for $u(x,0)$ the function above and solve for the Fourier cosine coefficients.

6. Consider the problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 12x^2$, $0 < x < 1, t > 0$, $\frac{\partial u}{\partial x}(0,t) = 1$, $u(1,t) = 0$.

a) Find the steady state solution $u_0(x)$.

b) Write $u(x,t) = u_0(x) + v(x,t)$ and find the pde and boundary conditions that v is to satisfy, if u is the solution to the given problem.

c) If v has the eigenfunction expansion $v(x,t) = \sum a_n e^{-\lambda_n t} \cos[(n-1/2)\pi x]$., then for large t about where will be the hottest spot along a rod with the heat distribution modeled by this problem?

1. Consider the problem $\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u & 0 < x < \pi, t > 0 \\ u(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t), & t > 0 \\ u(x,0) = 2\sin(3x/2), & 0 < x < \pi \end{cases}$

a) Separate variables, and derive the EVP and the time-dependent equation
b) Solve for the eigenvalues and associated eigenfunctions
c) Solve the t-equation, use the superposition principle to obtain the representation for $u(x,t)$.
d) Complete the problem by determining the Fourier coefficients and giving the final solution.

4. Consider the wave equation problem

$$\begin{cases} \frac{\partial}{\partial t}\{e^{-3x}\frac{\partial u}{\partial t}\} = \frac{\partial}{\partial x}\{e^{-3x}\frac{\partial u}{\partial x}\} & 0 < x < 1 \,,\ t > 0 \\ u(0,t) = 0 = u(1,t) & t > 0 \\ u(x,0) = f(x) \,,\ \frac{\partial u}{\partial t}(x,0) = g(x) & 0 < x < 1 \end{cases}$$

(a) Let $u(x,t) = T(t)\phi(x)$, separate variables and show that the eigenvalue problem is $\frac{d^2\phi}{dx^2} - 3\frac{d\phi}{dx} + \lambda\phi = 0$, $\phi(0) = 0 = \phi(1)$.

(b) Determine the eigenvalues and associated eigenfunctions for this problem. (do **not** finish of the problem further!)

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & 0 < x < \pi \,, \; t > 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 = u(\pi,t) & t > 0 \\ u(x,0) = \cos(x/2) \end{cases}$$

(a) Separate variables, and derive the EVP and the time-dependent equation.

(b) Solve for the eigenvalues and associated eigenfunctions. (All eigenvalues are positive in this problem, so need not check other cases.)

(c) Solve the t-dependent equation, and use superposition principle to write representation of $u(x,t)$ in terms of an eigenfunction series.

(d) Complete the problem by determining the Fourier coefficients, and give the final solution.

4. Consider the wave equation problem $e^{-x}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial}{\partial x}\left(e^{-x}\dfrac{\partial u}{\partial x}\right)$, $0 < x < 1, \quad t > 0$

$$u(0,t) = 0 = u(1,t), \quad t > 0$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 < x < 1$$

a) Let $u(x,t) = T(t)\varphi(x)$, and separate variables to determine the equation for $T(t)$ and the eigenvalue problem for $\varphi(x)$.

b) Determine the eigenvalues and associated eigenfunctions for this problem.

1. Consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} & 0 < x < 2\pi \,, \; t > 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(2\pi,t) & t > 0 \\ u(x,0) = \cos(10x) \,, \; u_t(x,0) \equiv 0 & 0 < x < 2\pi \end{cases}$$

When we separate variables and solve the eigenvalue problem, we find $\lambda = 0$ is an eigenvalue (with constant associated eigenfunction), and the other eigenvalues are $\lambda_n = n^2/4$, $n = 1, 2, \ldots$, with associated eigenfunctions being $\phi_n(x) = \cos(\frac{nx}{2})$.

(a) Solve for the $T(t) = T_n(t)$ and write out the resulting series for $u(x,t)$.

(b) Use the initial displacement and velocity functions to find the coefficients and write the final solution for $u(x,t)$. (Hint: rather than integrate, you can match coefficients to determine them.)

2. Consider the problem $\dfrac{\partial u}{\partial t} = 10^{-2}\dfrac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < \pi, \quad t > 0$

$$\frac{\partial u}{\partial x}(0,t) = 0 = u(\pi,t) \quad t > 0$$

$$u(x,0) = 2\cos(5x/2) - \cos(9x/2) \quad x > 0$$

a) If $u(x,t) = T(t)\varphi(x)$, derive the equation for T and the eigenvalue problem for $\varphi(x)$ (equation and boundary conditions)

b) Solve the eigenvalue problem to obtain the eigenvalues and associated eigenfunctions.

c) Use the initial data to determine the solution $u(x, t)$.

2. Consider the problem

$$\begin{array}{ll} u_t = 10^{-2} u_{xx} & 0 < x < \pi \,, \quad t > 0 \\ u_x(0,t) = 0 = u(\pi, t) & t > 0 \\ u(x,0) = 2\cos(5x/2) - \cos(9x/2) & 0 < x < \pi \end{array}$$

(a) If $u(x,t) = T(t)\phi(x)$, derive the equation for $T(t)$ and the eigenvalue problem for $\phi(x)$ (equation plus boundary conditions)

(b) Solve the eigenvalue problem to obtain the eigenvalues and associated eigenfunctions. Assume the fact that the only eigenvalues are positive reals. (Be sure you know what the eigenfunctions look like graphed.)

(c) Then solve the $T(t)$ equation, and then use the superposition principle to write the representation for $u(x,t)$. Do not bother to compute the Fourier coefficients.

Problem 4 ***(10pts)*** *Suppose $u = u(x,y,z)$ satisfies the Neumann problem*

$$\begin{aligned} \Delta u = 0, &\quad \textit{in } \Omega, \\ \vec{n} \cdot \operatorname{grad} u = 0, &\quad \textit{on } \partial\Omega. \end{aligned}$$

Show that u must be constant on Ω. (Hint: Green's first identity is

$$\int_\Omega u \Delta u \, dV = \int_\Omega \nabla u \cdot \nabla u \, dV + \int_{\partial\Omega} u \nabla u \cdot \vec{n} \, dA.)$$

solution:

4. Consider the wave equation problem

$$\begin{cases} \frac{\partial}{\partial t}\{e^{-5x}\frac{\partial u}{\partial t}\} = \frac{\partial}{\partial x}\{e^{-5x}\frac{\partial u}{\partial x}\} & 0 < x < 1 \,, \; t > 0 \\ u(0,t) = 0 = u(1,t) & t > 0 \\ u(x,0) = f(x) \,, \; \frac{\partial u}{\partial t}(x,0) = g(x) & 0 < x < 1 \end{cases}$$

(a) Let $u(x,t) = T(t)\phi(x)$, separate variables and put the equation for the eigenfunctions in Sturm-Liouville form. Then identify the weight function $\sigma(x)$, which is needed for the orthogonality property of the eigenfunctions.

(b) Show the eigenvalue problem can be written as $\frac{d^2\phi}{dx^2} - 5\frac{d\phi}{dx} + \lambda\phi = 0$, $\phi(0) = 0 = \phi(1)$, and determine the eigenvalues and associated eigenfunctions for this problem. (Do **not** finish of the problem further!)

4. Purely radial heat flow in 3-space satisfies

$$u_t = \frac{D}{r^2}\frac{\partial}{\partial r}\{r^2\frac{\partial u}{\partial r}\}, \quad \text{for } r = \sqrt{x^2+y^2+z^2} > 0,\ t > 0.$$

Write $u(r,t) = \frac{1}{r}v(r,t)$ and show that v satisfies the ordinary heat equation in terms of r and t. Then write out the general solution $u(r,t)$ to the equation.

7. **Laplace Equation** (8 pts) Solve the following
$\nabla^2 u = 0$ on $\{(x,y) : 0 < x < \pi, 0 < y < \pi\}$,
$u_y(x,0) = 0, u_y(x,\pi) = 0, u(0,y) = f(y), u(\pi,y) = 0.$

Problem 6 ***(10pts)*** *Let w be a scalar field and ϕ a vector field. Verify the vector identity*

$$\operatorname{div}(w\phi) = \phi \cdot \operatorname{grad} w + w \operatorname{div} \phi. \tag{1}$$

Integrate this equation over Ω and take $\phi = \operatorname{grad} u$, where u is a scalar field, to prove Green's identity

$$\int_\Omega w\Delta u\, dV = -\int_\Omega \operatorname{grad} u \cdot \operatorname{grad} w\, dV + \int_{\partial\Omega} w \operatorname{grad} u \cdot \mathbf{n}\, dA.$$

solution:

2. Consider the heat equation **problem** in a square:

$$\begin{cases} u_t = \nabla^2 u & \text{in } \Omega = \{(x,y) : 0 < x < \pi, 0 < y < \pi\} \\ u_x(0,y) = u_y(x,0) = u_y(x,\pi) = 0 & \text{and} \quad u(\pi,y) = 6 \\ u(x,y,0) = f(x,y) \end{cases}$$

Solve, via separation of variables method, only the steady-state problem, that is, the Laplace's equation problem.

3. Consider the Poisson problem

$$\begin{cases} \nabla^2 u = -e^{-x^2-3y^2} & \text{in } \Omega = \{(x,y) : y > -1, |x| < \infty\} \\ u(x,-1) = 0 & |x| < \infty \end{cases}$$

(a) Construct the Green's function using the method of reflection.

(b) Write out the solution $u(x,y)$ in Ω in terms of the Green's function, but do **not** try to integrate the expression. (Be very careful with limits of integration.)

3. Consider the Poisson problem

$$\begin{cases} \nabla^2 u = -e^{-y^2} & \text{in } \Omega = \{(x,y) : y > -1, |x| < \infty\} \\ u(x,-1) = 0 & |x| < \infty \end{cases}$$

(a) Construct the Green's function using the method of reflection.

(b) Write out the solution $u(x,y)$ in Ω in terms of the Green's function, but do **not** try to integrate the expression. (Be very careful with limits of integration.)

2. Consider the problem for Laplace's equation in a wedge:

$$\begin{cases} \nabla^2 u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{in } \Omega = \{(r,\theta) : 0 \le r < 1, 0 < \theta < \frac{\pi}{4}\} \\ u(r,0) = 0 = u(r,\pi/4) & 0 \le r < 1 \\ u(1,\theta) = 3\sin(12\theta) & 0 \le \theta \le \pi/4 \\ u \text{ remains bounded as } r \to 0. \end{cases}$$

(a) By separation of variables method, $u(r,\theta) = R(r)\Theta(\theta)$, determine the equations for $R(r)$ and $\Theta(\theta)$.

(b) Determine the eigenvalues and associated eigenfunctions for the Θ equation and its boundary conditions.

(c) Solve the $R(r)$ equation.

(d) Use superposition principle to write the representation for solution $u(r,\theta)$.

(e) Determine the coefficients and write the final expression for u.

2. Consider the problem for Laplace's equation in a wedge:

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 \le r < 1, 0 < \theta < \pi/2\} \\ u(r,0) = 0 = \frac{\partial u}{\partial \theta}(r,\pi/2) & 0 \le r < 1 \\ u(1,\theta) = \sin(7\theta) & 0 \le \theta \le \pi/2 \\ u \text{ remains bounded as } r \to 0. \end{cases}$$

Remember that

$$\nabla^2 u := \frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial u}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(a) By separation of variables method, $u(r,\theta) = R(r)\Theta(\theta)$, determine the equations for $R(r)$ and $\Theta(\theta)$.

(b) Determine the eigenvalues and associated eigenfunctions for the Θ equation and its boundary conditions.

(c) Solve the $R(r)$ equation.

(d) Use superposition principle to write the representation for solution $u(r,\theta)$.

(e) Determine the coefficients and write the final expression for u.

2. Consider Laplace's equation in a wedge:

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 \le r < 1, 0 < \theta < \pi/2\} \\ \frac{\partial u}{\partial \theta}(r,0) = 0 = u(r,\pi/2) & 0 \le r < 1 \\ u(1,\theta) = 2\cos(3\theta) & 0 \le \theta \le \pi/2 \\ u \text{ remains bounded as } r \to 0 & \end{cases}$$

(a) By separation of variables method, $u(r,\theta) = R(r)\Theta(\theta)$, determine the equations for $R(r)$ and $\Theta(\theta)$.

(b) Determine the eigenvalues and associated eigenfunctions for the Θ equation and its boundary conditions.

(c) Solve the $R(r)$ equation.

(d) Use superposition principle to write the representation for solution $u(r,\theta)$.

(e) Determine the coefficients and write the final expression for u.

4. Consider the Poisson problem

$$\begin{cases} \nabla^2 u = -1 & \text{in } \Omega = \{(x,y) : y < 1, |x| < \infty\} \\ u = 0 & \text{on } \partial\Omega = \{(x,y) : y = 1, |x| < \infty\} \end{cases}$$

Construct the Green's function using the method of reflection.

5. For Laplace's equation in a wedge be: $\nabla^2 u = 0$ in $\Omega = \{(r,\theta) \mid 0 \le r < 1, 0 < \theta < 3\pi/4\}$

$\frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}(r,\frac{3\pi}{4}) = 0,\ 0 \le r < 1,\ u(1,\theta) = f(\theta),\quad 0 < \theta < \pi/4$, where u remains bounded as $r \to 0$, do the following:

a) By separation of variables method, $u(r,\theta) = R(r)\Theta(\theta)$, determine the equation for $R(r)$ and the eigenvalue problem for $\Theta(\theta)$.

b) Determine the eigenvalues and associated eigenfunctions from the $\Theta(\theta)$ equation.

c) Then solve the $R(r)$ equation.

d) Use superposition principle to write the representation for $u(r,\theta)$. What is the solution if $f(\theta) \equiv 3$?

2. Consider the problem for Laplace's equation in a wedge:

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = \{(r,\theta) : 0 \leq r < 1, 0 < \theta < \pi\} \\ u(r,0) = 0 = \frac{\partial u}{\partial \theta}(r,\pi) & 0 \leq r < 1 \\ u(1,\theta) = 6\sin(3\theta/2) & 0 \leq \theta \leq \pi \\ u \text{ remains bounded as } r \to 0. \end{cases}$$

Remember that

$$\nabla^2 u := \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\frac{\partial u}{\partial r} \right\} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(a) By separation of variables method, $u(r,\theta) = R(r)\Theta(\theta)$, determine the equations for $R(r)$ and $\Theta(\theta)$.

(b) Determine the eigenvalues and associated eigenfunctions for the Θ equation and its boundary conditions.

(c) Solve the $R(r)$ equation.

(d) Use superposition principle to write the representation for solution $u(r,\theta)$.

(e) Determine the coefficients and write the final expression for u.

4. Consider the Poisson problem

$$\begin{cases} \nabla^2 u = -e^{-x^2} & \text{in } \Omega = \{(x,y) : y < -1, |x| < \infty\} \\ u(x,-1) = 0 & |x| < \infty \end{cases}$$

(a) Construct the Green's function using the method of reflection.

(b) Write out the solution $u(x,y)$ in Ω in terms of the Green's function.