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Motivating Characteristics—Solutions

MATH 404, Fall 2021

In this activity, we'll consider the first order *transport* PDE:

$$u_t + cu_x = f, \quad x \in \mathbb{R}, \ t \geq 0. \tag{1}$$

Above, $u(x,t)$ is a scalar-valued (unknown) function that depends on both time and space. We purposefully leave the dependencies of c and f vague—note that both could depend on any (or all) of: x, t, u.

1. We have discovered that $u(x,t) = u_0(x - ct)$ solves the IVP:

$$\begin{cases} u_t + cu_x = 0 \\ u(x,0) = u_0(x). \end{cases}$$

Suppose that $u_0(x) = e^{-x^2}$. Describe in a sentence (or draw a picture describing) what the solution "does"; i.e., how do the dynamics evolve $u_0(x) \mapsto u(x,t)$? What does the "movie" look like?

Solution: Recall that subtracting from the argument of a function "translates" the graph of the function to the right. Hence taking a function f and considering $f(x-ct)$ this is a constant speed (speed $= c$) translation of the graph of f to the right. In one unit of time, the entire graph $u_0(x) = e^{-x^2}$ moves to the right c units.

2. Now, consider $x(t)$ to be some function for $t \geq 0$. Compute (using the multivariable chain rule):

$$\frac{d}{dt}\Big[u\big(x(t),t\big)\Big] = u_x(x(t),t)\frac{dx}{dt}(x(t),t) + u_t(x(t),t).$$

Solution: This can be written in Leibniz form as: $\frac{d}{dt}\Big[u\big(x(t),t\big)\Big] = \Big[\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\Big]\Big|_{(x(t),t)}$.

3. Compare your calculation above to the LHS of equation (1). Produce an *ODE* in $x(t)$ that will force the two expressions to match. Give your ODE the IC: $x(0) = \xi$ (this variable is "xi".)

Solution: In comparing $\frac{d}{dt}\Big[u\big(x(t),t\big)\Big] = u_x(x(t),t)\frac{dx}{dt}(x(t),t) + u_t(x(t),t)$ and $u_t + cu_x$, we want to match $\frac{dx}{dt}$ with c. This yields the IVP:

$$\begin{cases} \dfrac{dx}{dt} = c \\ x(0) = \xi \end{cases}$$

4. Suppose $x(t)$ solves your *characteristic ODE* from the previous part. Consider the equation:

$$\frac{d}{dt}\left[u(x(t),t)\right] = f(x(t),t).$$

What type of equation is this (how many independent variables are there)? If you solve this equation, what have you actually done (at least restricting to $x = x(t)$)?

Solution: This equation is an ODE, since by assuming $x = x(t)$ we have reduced the entire problem to the independent variable t.

By the previous parts, if we solve this ODE, we have actually solved the ODE:

$$u_t(x(t),t) + \frac{dx}{dt}(x(t),t)u_x(x(t),t) = f(x(t),t).$$

But, if we have solved the characteristic ODE, we know that $\frac{dx}{dt} = c$, so if we solve these ODEs, we have solved the PDE *along the trajectory* $x(t)$.

5. Let $c(x,t)$ and $f(x,t)$ be given/known functions. Conjecture about how to solve the problem

$$u_t + c(x,t)u_x = f(x,t), \quad x \in \mathbb{R}, \ t \geq 0. \tag{2}$$

by eliminating an independent variable and turning the PDE into two different (solvable) ODEs. (Write two or three sentences.)

Solution: First introduce the characteristic ODE, then solve it. After solving it, consider the ODE $\frac{d}{dt}[u(x(t),t)] = f(x(t),t)$. With these solutions, we have solved the PDE...but with the exception that we have to consider the curve $x(t)$—this may not be an arbitrary x, since $x = x(t)$. We will resolve this issue through "inversion".