Lecture 2: Terminology and Classification

Math 404

9/3/25

Warm Up: Fluids

Eulerian framework: $\vec{v}(\vec{x}, t)$ describes the flow field at $(\vec{x}, t)$

Lagrangian framework: $\vec{X}(\vec{x}_0, t)$ describes position at time t emanating from $\vec{x}_0$

(Draw Picture)

Defining relation: $\vec{v}\left(\vec{X}(\vec{x}_0, t), t\right) = \frac{d}{dt}\vec{X}(\vec{x}_0, t)$

Let $\theta(x, t)$ be temp. in Eulerian coordinates.

How does θ change along a flow line?

What is $\frac{d}{dt}\left[\theta\left(\vec{X}(\vec{x}_0, t), t\right)\right]$?

PDEs, IVPs, BVPs, IBVPs

Evolutions

Solution (typically) $u(t, \vec{x})$

$$u_t + L(\vec{x}, u, D^\alpha u) = F \qquad \text{or} \qquad u_{tt} + L(\vec{x}, u, D^\alpha u) = F.$$

Initial conditions:
$u(t=0) = u_0$ or $u(t=0) = u_0(\vec{x}), \; u_t(t=0) = u_1(\vec{x})$.

Stationary Problems

Solution (typically) $u(\vec{x})$

$$L(\vec{x}, u, D^\alpha u) = F.$$

Boundary conditions...specified on the boundary!

Examples I

$$\begin{cases} T_x + T_t = f(x) & \text{in} \quad \mathbb{R} \\ T(t=0) = \begin{cases} x-1 & [-1,0) \\ 1-x & [0,1) \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

$$\begin{cases} u_{tt} + ku_t = u_{xx} + f(u) & \text{in} \quad [0,\infty) \\ \partial_x u(x=0) = 0 & \partial\mathbb{R}_+ = \{0\} \\ u(t=0) = e^{-x^2}; \quad u_t(t=0) = 0 \end{cases}$$

Examples II

$$\begin{cases} \theta_t = \operatorname{div}\big(k(x,y,z)\nabla\theta\big) & \text{in} \quad \Omega \subset \mathbb{R}^3 \\ \theta\Big|_\Gamma = \theta_0 + F(x,y,z) & \text{on} \quad \Gamma = \partial\Omega \\ \theta(t=0) = \theta_0(x,y,z) \end{cases}$$

$$\begin{cases} z_{xx} + z_{yy} = 0 & \text{in} \quad \mathbb{D} \\ z = 0 & \text{on} \quad \{(x,y) \ : \ x^2 + y^2 = 1\} \end{cases}$$

Well-Posedness...

An IVP/IBVP is *well-posed* (in the sense of Hadamard) if...

- A solution exists
- The solution is unique
- (*) The solution depends continuously (in some sense) on the data in the problem.

PDEs corresponding to physical phenomena should (typically) be well-posed.

And A Solution

Consider the IBVP in $w : (0, \pi) \times [0, T] \to \mathbb{R}$:

$$\begin{cases} w_{tt} + w_{xxxx} = 0 \\ w(x = 0) = w_{xx}(x = 0) = w(x = \pi) = w_{xx}(x = \pi) = 0 \\ w(t = 0) = 0; \quad w_t(t = 0) = \sin(x) \end{cases}$$

What about $w(x, t) = \sin(x) \sin(t)$?

(simulation, if time)

Linear and Nonlinear

An operator L is linear if for all a, b scalars and f, g functions

$$L[af + bg] = aL[f] + bL[g].$$

Examples?

An PDE in the variable u is linear if it is linear in u and its derivatives:
$\sum a_\alpha(\vec{x}, t) D^\alpha u = F(\vec{x}, t).$

An equation is nonlinear if it is... not linear.

This means: interactions of terms involving u and its derivatives $D^i u$.

Nonlinearity

There are classifications of nonlinear equations:

- **Semilinear**: the equation is linear in the highest order terms: $a(\vec{x}, t)D^n u + LOT = 0$
- **Quasilinear**: the equation is nonlinear in the highest order terms, but only in the sense of coefficients depending on lower order terms: $L(\vec{x}, t, u, Du, ..., D^{n-1}u)D^n u + LOT = 0.$
- **Fully Nonlinear**: the equation is nonlinear in its highest order term: $L(\vec{x}, t, u, Du, ..., D^n u) = 0.$

Let's Create Some Examples

First and Second Order:

Linear PDE

Semilinear PDE

Quasilinear PDE

Fully nonlinear PDE

Linear: Superposition and Differences

Consider two solutions u_1, u_2 to a homogeneous, **linear** PDE/BVP:

Abstractly:

$$L[u_1] = 0, \quad L[u_2] = 0$$

What about:

$$L[cu_1] = \ldots \qquad L[u_1 + u_2] = \ldots$$

In general: $L[c_1 u_1 + c_2 u_2] = \ldots$

Differences for inhomogeneous:

$$L[u_1] = F, \quad L[u_2] = F$$

$$L[f - g] = L[f] - L[g] = 0$$

(Homogeneous wave)

Second Order Linear I

General linear second order PDE in two independent variables (say $\vec{x} = \langle x_1, x_2 \rangle$):

$$A(\vec{x})\partial_1^2 u + 2B(\vec{x})\partial_1\partial_2 u + C(\vec{x})\partial_2^2 u + d(\vec{x})\partial_1 u + e(\vec{x})\partial_2 u + f(\vec{x})u + g(\vec{x}) = 0$$

Rewrite as:

$$Au_{x_1x_1} + 2Bu_{x_1x_2} + Cu_{x_2x_2} + LOT = 0$$

$$\text{sign of} \quad B^2 - AC$$

$$A(\vec{x}), B(\vec{x}), C(\vec{x})$$

Sign can depend on $\vec{x}$. Change of type possible.

Second Order Linear II

In any $\vec{x}$ region where:

$$B^2 - AC > 0 \qquad \text{equation/operator is } \textbf{hyperbolic}$$

$$B^2 - AC < 0 \qquad \text{equation/operator is } \textbf{elliptic}$$

$$B^2 - AC = 0 \qquad \text{equation/operator is } \textbf{parabolic}$$

Wave, Laplace, Heat $\quad (t, x), \quad (x, y), \quad (t, s)$

$$Au_{x_1x_1} + 2Bu_{x_1x_2} + Cu_{x_2x_2} = [\partial_{x_1}, \partial_{x_2}] \begin{bmatrix} A(x_1, x_2) & B(x_1, x_2) \\ B(x_1, x_2) & C(x_1, x_2) \end{bmatrix} \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \end{bmatrix} u$$

More variables?

DON'T FORGET THAT 2 ABOVE!

Tricomi's equation: $\quad u_{xx} - xu_{yy} = 0.$

PDEs are Hard

Why CLASSIFY?

Tehniques, Tools, and Qualitative Properties

PDEs are Hard

Solving versus simulating; getting our perspective straight

Often we do not “solve” PDEs in the traditional calculus/ODE sense (though we will do some of this here)

$$\begin{cases} u_{tt} - D_1 \left[u_x + \frac{1}{2}(w_x)^2\right]_x = 0 \\ (1-\alpha\partial_x^2)w_{tt} + D_2\partial_x^4 w + k_0(1-\alpha\partial_x^2)w_t - D_1\left[w_x(u_x + \frac{1}{2}w_x^2)\right]_x = p(x,t) \\ u(0) = 0; \quad D_1\left[u_x(L) + \frac{1}{2}w_x^2(L)\right] = 0 \\ w(0) = w_x(0) = 0; \quad D_2 w_{xx}(L) = 0; \qquad -\alpha\partial_x[w_{tt}(L) + k_0 w_t] + D_2 w_{xxx}(L) = 0. \end{cases}$$

(simulation)