# Lecture 1: Intro to PDE

Math 404

8/27/25

# Review Material

FUNCTIONS

$$f:\ \mathbb{R}\to\mathbb{R} \qquad f:\ \mathbb{R}^n\to\mathbb{R} \qquad \vec{f}:\ \mathbb{R}\to\mathbb{R}^m \qquad \vec{F}:\ \mathbb{R}^n\to\mathbb{R}^m$$

Examples?

Vector notation: $\vec{v}$, $v_i$, $\vec{v}\cdot\vec{w}$, $v_i v_i$

Notions of *change*: $\partial_i$, $\nabla$, $D_{\vec{v}}$, div, curl, $\Delta$

Notions of *integration*: $\int_a^b dx$, $\iint_\Omega dA$, $\iiint_O dV$, $\oint ds$, $\oiint dS$

Sequences, series, and associated notation: $a_n$, $f_n(x)$, $\sum_i a_i$, $\sum_{n=1}^{\infty} f_n(x)$

**Calculus II and III, ODE**

HW1—http://webster.math.umbc.edu/HW1&.pdf
GET STARTED!

# Modeling and PDEs

Partial Differential Equations (versus *ordinary* differential equations) are the language of physical sciences (*continuum* phenomena)

Mathematical modeling of phenomena:
Newton's second law, conservation of energy, principle of virtual work, Hamilton's principle, principle of least action

A branch of mathematics unto itself

PDEs do arise in application to other pure math fields; $L(x, y, u, D^\alpha) = 0$

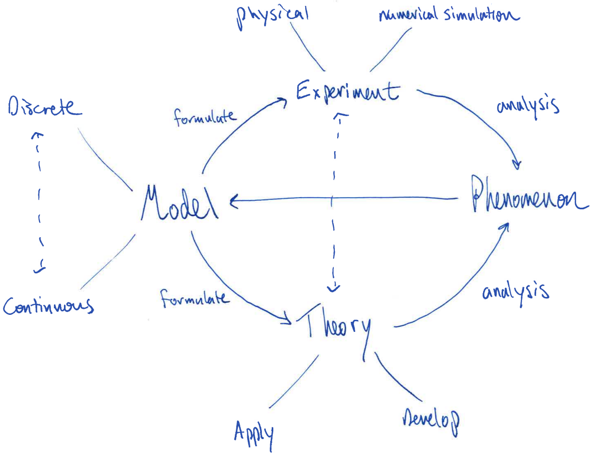
Discrete versus continuous

$$\int_a^b f(x)dx \quad \leftrightarrow \quad \sum_{i=1}^{N} f(x_i^*)\Delta x; \qquad \left.\frac{df}{dx}\right|_{x_0} \quad \leftrightarrow \quad \frac{f(x_0+h)-f(x_0)}{h}$$

Limiting procedures

# Applied Mathematics: ~~The~~ A Big Picture

+ *math motivated by a problem in the "real world"*
+ studying *phenomena of interest* using mathematical models
+ *developing new theory* or *applying existing theory* in doing so



# Modeling II

*...all models are approximations. Essentially,* **all models are wrong, but some are useful.** *However, the approximate nature of the model must always be borne in mind....*

—George Box, *Empirical Model-Building*, 1987

**A map is not the territory it represents**, *but, if correct, it has a similar structure to the territory, which accounts for its usefulness.*

—Alfred Korzybski, *Science and Sanity*, 1933

# Motivational Examples

$x(t)$ a real-valued function

$$mx''(t) + dx'(t) + kx(t) = 0$$

$u(x,t)$ is a real valued function

$$u_{tt}(x,t) + du_t(x,t) - c^2 u_{xx}(x,t) = 0$$

Different notions of *change* in the same equation; *evolution* of states along one variable

# Some Famous PDEs and Some Terminology

$z(x, t)$ a real-valued function

$$z_t + a(x, t)z_x = f(x, t)$$

$\theta(\vec{x}, t)$ a real-valued function

$$\theta_t = K(\vec{x})\Delta\theta$$

$\Psi(\vec{x}, t)$ a complex-valued function

$$\Psi_t = i\hbar\Delta\Psi + V(\Psi)$$

$\vec{v}(\vec{x}, t)$ a vector-valued function, $p(\vec{x}, t)$ a real-valued function

$$\vec{v}_t - \nu\Delta\vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = 0, \quad \nabla \cdot \vec{v} = 0$$

$w(x, t)$ a real-valued function

$$w_{tt} + Dw_{xxxx} = f(w)$$

$u(\vec{x}, t)$ a real-valued function

$$-\Delta u = F$$

# Domains and Boundaries II

PDEs and boundary conditions, example:

$$\begin{cases} \Delta u(x,y) = -\lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \Gamma \end{cases}$$

$$\begin{cases} u_{xx} + u_{yy} + \epsilon u = F(x,y) & \text{in} \quad \Omega \\ \nabla u \cdot \vec{n} = 0 & \text{on} \quad \Gamma \end{cases}$$

$C^{\alpha}(\Omega)$ spaces; example: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

# Differential Operators and Terms I

General notation: $\partial_i, \quad \partial_{x_i}, \quad \partial_x, \quad D, \quad D^\alpha$

Operator $L$: $\quad L(x), \quad L(x, \partial_x), \quad L(x, \partial_x, u), \ldots$

$$L[f] = af'' + bf' + cf, \quad L(x)[f] = a(x)f'(x),$$

$$L(f) = a(x)f^2(x), \qquad L(x, f) = x\sin(f(x))$$

$$Af = -\Delta f, \qquad A(\vec{x})f = -\sum_{i,j}^{n} a_{ij}(\vec{x})\partial_{x_i}\partial_{x_j} f(x_1, \ldots, x_n).$$

A PDE/BVP is **homogeneous** if all terms depend on the solution variable.

**Order**: The highest order derivative present; sometimes we talk about spatial order versus temporal order.

**Coefficients**: Things in front of differential operators.

**Data**: Boundary, Initial, Inhomogeneous terms

# Differential Operators and Terms II

$$\partial_x u = 0$$

$$\partial_x u + \partial_t u = 0$$

$$\partial_x u + \partial_t u = f(x)$$

$$a(x)\partial_x u + \partial_t u = f(x, t)$$

$$\partial_x u + \partial_t u = f(u)$$

$$g(u)\partial_x u + \partial_t u = f(x)$$

$$G(\partial_x u) + \partial_t u = 0$$

# Application: Aeroelastic Instability of Suspension Bridges

Tacoma Narrows Bridge Catastrophe, 1940; "Galloping Gertie"

▸ Longitudinal ▸ Torsional ▸ Collapse

# Research Question

**What are the mechanisms of bridge flutter?**

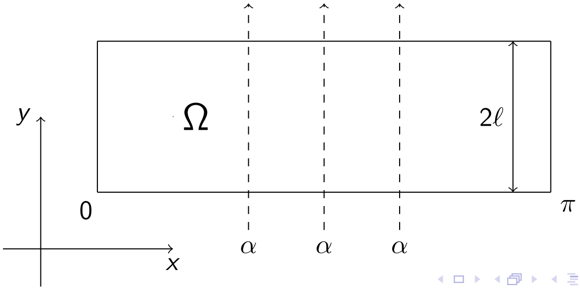
*Choose an aeroelastic model and:*

- determine what causes flutter behaviors to shift from longitudinal to torsional
- use analytical/numerical methods to predict (for this model) for what parameters this shift will occur
- explicitly construct periodic solutions for this model

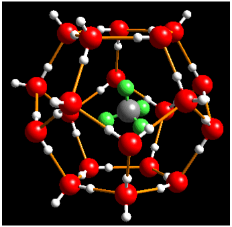
# Bridge Plate Model with Simple Aerodynamic Loading

$$\begin{cases} w_{tt} + kw_t + \Delta^2 w + \left[P - S \int_\Omega w_x^2\right] w_{xx} = p_0(x,y) + \alpha w_y & \text{in } \Omega \times (0,T) \\ w = w_{xx} = 0 & \text{on } \{0,\pi\} \times [-\ell,\ell] \\ w_{yy} + \sigma w_{xx} = w_{yyy} + (2-\sigma) w_{xxy} = 0 & \text{on } [0,\pi] \times \{-\ell,\ell\} \\ w(x,y,0) = w_0(x,y), \qquad w_t(x,y,0) = v_0(x,y) & \text{in } \Omega. \end{cases}$$

where $\Delta w = w_{xx} + w_{yy}$, so $\Delta^2 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}$



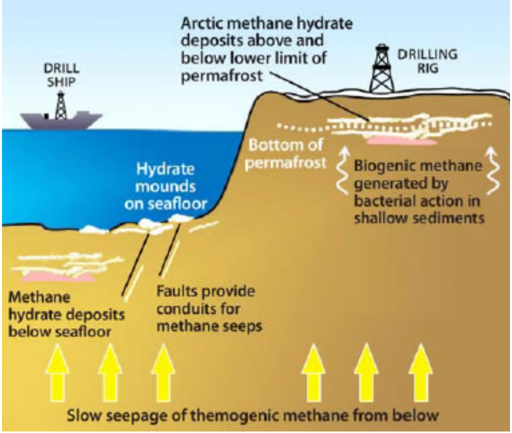
## Methane Hydrates (Clathrates)




Hydrates

Image Credit: USGS and NETL

# Hydrate Regions and Stability



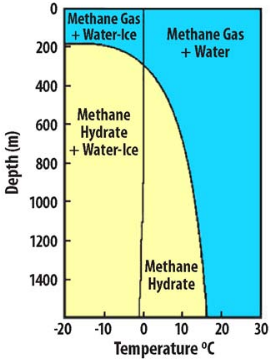
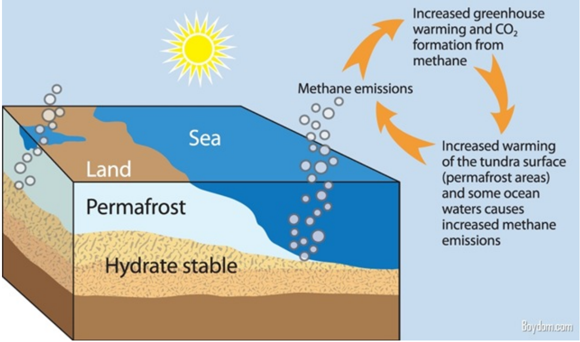




Image Credit: NETL

# Methane Hydrates: Fuel versus Feedback

Natural Gas (energy, GOOD) vs. Global Climate Instability (very bad)

Greenhouse effect and greenhouse gasses




*Feedback mechanisms*: microphone squealing!

# Hydrate Formation: Conservation of Mass for Methane

**Concentration:** (mass of methane per unit volume per unit time)

$$\partial_t \left[\phi \left(S_l \rho_l \chi_l^M + S_h \rho_h \chi_h^M\right)\right] + \nabla \cdot \left[\rho_l \chi_l^M \mathbf{q} - \phi S_l \rho_l D_0 \nabla \chi_l^M\right] = f_M \tag{1}$$

Maximum solubility constraint: $\chi_{l,max}^M(p, \theta, \chi_l^S) \approx \chi_{l,max}^M(x)$.

Thermodynamic (Gibbs phase rule) relation:

$$\begin{cases} \chi_l^M \leq \chi_{l,max}^M(x); \quad S_l = 1 \\ \chi_l^M = \chi_{l,max}^M(x); \quad S_l \leq 1 \\ (\chi_{l,max}^M(x) - \chi_l^M) \cdot (1 - S_l) = 0 \end{cases} \tag{2}$$

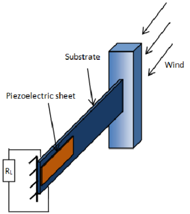
General Model: $\Omega \subset \mathbb{R}^N, \quad N = 1, 2, 3, \ \partial\Omega$ smooth

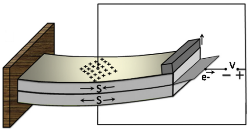
$$\begin{cases} u_t - D\Delta v + \nabla \cdot [v\mathbf{q}] = f \\ (v, u) \in \beta(x; \cdot) \equiv \{(v, v) : \ v \leq v^*(x)\} \cup \{v^*(x)\} \times [v^*(x), R) \\ v = 0 \text{ on } \ \partial\Omega \end{cases} \tag{3}$$

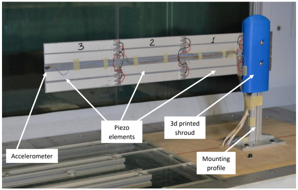
# Application: Piezoelectric Energy Harvesting

**Goal**: understand *axial cantilever flutter* for *energy harvesting*

- Remote sources of power
- Harvest from ambient, **low-velocity** flows
- Reduction of battery dependence




▸ Bal Experiment

**Problem:** *capturing and predicting nonlinear large deflections of a cantilever (LCOs) for optimization*

# Recent Result—Baseline Theory of Solutions

$$\begin{cases} w_{tt} + D\partial_x^4 w - D\partial_x[w_{xx}^2 w_x] + D\partial_x^2[w_x^2 w_{xx}] + \partial_x[w_x \int_x^L u_{tt}] = p(x,t) \\ u_{tt}(x) = -\int_0^x \left[w_{xt}^2 + w_x w_{xtt}\right] d\xi \\ w(0) = w_x(0) = 0; \quad w_{xx}(L) = w_{xxx}(L) = 0 \\ w(0) = w_0, \quad w_t(0) = v_0 \end{cases}$$

## Theorem (Deliyianni and W., *AMO*, 2020)

*Let $p,\ p_t,\ p_{xx} \in L^2(0,T;L^2(0,L))$. For smooth data $w_0 \in \mathcal{D}(\mathcal{A}^2)$, $w_1 \in \mathcal{D}(\mathcal{A})$, strong solutions exist up to some time $T^*(w_0,w_1)$. For all $t \in [0,T^*)$, the solution $w$ is* unique *and obeys the energy identity:*

$$E(t) + D_0^t = E(0) + \int_0^t (p, w_t)d\tau.$$

$$\mathcal{D}(\mathcal{A}) = \left\{w \in H^4(0,L) : w(0) = w_x(0) = 0,\ w_{xx}(L) = w_{xxx}(L) = 0\right\}$$

# Application: Ocular Dynamics

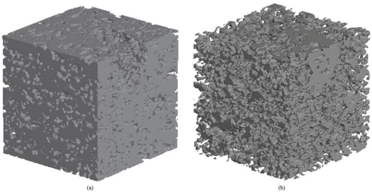
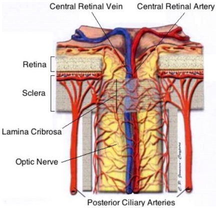
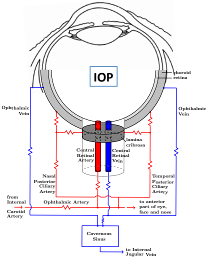



Figure 10. 3D images (401 μm³) reconstructed for ceramic I sample a) solid phase; and b) porous phase.




**Goal 1**: Understand the dynamics of the eye (lamina cribrosa) in response to daily pressure changes

**Goal 2**: Attempt to use model (and analysis thereof) to say something about glaucoma onset

# Biot's Equations of Poro-elasticity for Tissue Modeling

$$
\begin{aligned}
-\mu_e \Delta \mathbf{u} - (\lambda_e + \mu_e)\nabla(\nabla \cdot \mathbf{u}) + \mu_v \Delta \mathbf{u}_t - (\lambda_v + \mu_v)\nabla(\nabla \cdot \mathbf{u}_t) + \alpha \nabla p = \mathbf{F} \quad & \text{in} \quad \Omega \times (0, T) \\
(\alpha \nabla \cdot \mathbf{u})_t - \nabla \cdot (k(\nabla \cdot \mathbf{u}) \nabla p) = S \quad & \text{in} \quad \Omega \times (0, T) \\
\nabla \cdot \mathbf{u} = d_0 \quad & \text{in} \quad \Omega, \ t = 0 \\
\mathbf{T}(\mathbf{u}, p)\, \mathbf{n} = \mathbf{g} \quad & \text{on} \quad \Gamma_N \times (0, T) \\
\mathbf{u} = \mathbf{0} \quad & \text{on} \quad \Gamma_D \times (0, T) \\
\nabla p \cdot \mathbf{n} = 0 \quad & \text{on} \quad \Gamma_N \times (0, T) \\
p = 0 \quad & \text{on} \quad \Gamma_{D,p} \times (0, T \\
-k(\nabla \cdot \mathbf{u}) \nabla p \cdot \mathbf{n} = \psi \quad & \text{on} \quad \Gamma_{D,v} \times (0, T
\end{aligned}
$$

**Key Observation**: The imbalance of elastin and collagen, due to natural collagen degradation, leads to less regular responses to stark changes in IOP.