April 25, 2023

Discussion of the Neumann Laplace Problem on the Half Space

MATH 404, Spring 2023

In your HW you were asked the following problem (it is 4(c)):

(Bell p. 5–6; DuZ pp.207–209) *Consider Laplace's equation on the half space*

$$\mathbb{H} \equiv \{(x, y) \in \mathbb{R}^2 \ : \ y > 0\}$$

for the solution variable $u(x, y)$ with Neumann data:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 & \textit{for} \quad (x, y) \in \mathbb{H} \\ u_y(x, 0) = g(x). \end{cases}$$

Solve the BVP by taking the Fourier transform in x and enforcing the condition that $u(x, y)$ stays bounded for all $(x, y) \in \mathbb{H}$. (Be careful about constants of integration.)

I will discuss the above problem, but first, I want to discuss a modified problem:

$$\begin{cases} u_{xx} + u_{yy} - u = 0 & \text{for} \quad (x, y) \in \mathbb{H} \\ u_y(x, 0) = g(x). \end{cases} \tag{1}$$

Solution: Let us take the Fourier transform of the equation and the data equality in x, which is to say, apply $\mathscr{F}_x$ to the entire problem in order to obtain the Cauchy problem (ODE):

$$\begin{cases} -\xi^2 \hat{u} + \hat{u}_{yy} - \hat{u} = 0 & \text{for} \quad (\xi, y) \in \hat{\mathbb{H}} \\ \hat{u}_y(\xi, 0) = \hat{g}(\xi). \end{cases}$$

where we recall that $\widehat{u_{xx}} = (-i\xi)^2 \hat{u}$, using the rule for derivatives.

So, simplifying, we need to solve the ODE in y:

$$\hat{u}_{yy} - (1 + \xi^2)\hat{u} = 0.$$

Using our bountiful knowledge of ODE theory, we have a general solution of the form:

$$\hat{u}(\xi, y) = c_1 e^{-y\sqrt{1+\xi^2}} + c_2 e^{y\sqrt{1+\xi^2}}.$$

So we know that we will ultimately need to invert the Fourier transform, we hope our transform $\hat{u}$ will be in $L^2_\xi(\mathbb{R})$ for each y to ensure that we can invert. If we retain the term involving c_2 (since it is unbounded in ξ), we will have no shot at inversion. Thus we *require* (essentially as an auxiliary condition) that $\hat{u}(\xi, y) \to 0$ as $\xi \to \infty$. (This will be true if $u \in L^2$ in x, or even if $u \in L^1$ in x.) Thus we take $c_2 = 0$.

We now invoke the boundary condition to solve for c_1, recalling that $\hat{u}_y(\xi, 0) = \hat{g}(\xi)$. So differentiate (of course, noting that the solution is "frozen" in ξ when we differentiation in y), to obtain:

$$\hat{g}(\xi) = \hat{u}_y(\xi, y) = -c_1\sqrt{1 + \xi^2}\exp(-y\sqrt{1 + \xi^2}).$$

Set $y = 0$ to solve and obtain $c_1 = \dfrac{-g(\xi)}{\sqrt{1 + \xi^2}}$, and thus we have our transform (yay!);

$$\hat{u}(\xi, y) = \frac{-g(\xi)}{\sqrt{1 + \xi^2}}\exp(-y\sqrt{1 + \xi^2}).$$

With a mind toward the convolution rule, let us rewrite this suggestively:

$$\hat{u}(\xi, y) = \hat{g}(\xi)\frac{-\exp(-y\sqrt{1+\xi^2})}{\sqrt{1+\xi^2}}.$$

We need to invert, and we have a pointwise product in the Fourier (dual) space. Thus we know that the solution will be convolution with the data:

$$u(x, y) = \mathscr{F}_x^{-1}[\hat{g}] *_x \mathscr{F}_x^{-1}\left[\frac{-\exp(-y\sqrt{1+\xi^2})}{\sqrt{1+\xi^2}}\right].$$

It is clear how to invert $\hat{g}$ (think "isomorphism") so long as $g \in L^2(\mathbb{R})$, but it is by no means a straightforward matter to invert the second function. We opt for a different approach; let us (well, you) first *prove* that this function can be inverted under $\mathscr{F}$. After doing so, we will simply **define** a function (which we will know exists) **as the inverse** transform.

Let us define a function $j : \mathbb{R} \to \mathbb{R}$ of a dummy variable z, and let $y \in \mathbb{R}_+$ be "frozen":

$$j(z) = \frac{-\exp(-y\sqrt{1+z^2})}{\sqrt{1+z^2}}.$$

You should be able to prove that $j \in L^2(\mathbb{R})$; note, you won't be able to explicitly integrate it (with a specific known value), but you should be able to argue THAT j^2 is integrable over $\mathbb{R}$—this is not straightforward, but try.

In any case, $j \in L^2(\mathbb{R})$. And thus we can define a function

$$G(x, y) = \mathscr{F}_\xi^{-1}\left[\frac{-\exp(-y\sqrt{1+\xi^2})}{\sqrt{1+\xi^2}}\right](x).$$

We didn't just choose G as the name for no reason; this G is in fact the *Green's function* for this problem.

To solve the BVP given in (1), simply take your data $g \in L^2(\mathbb{R})$ and *convolve it in x with the Green's function*! Thus

$$u(x, y) = g(x, y) *_x G(x, y).$$

If you chose to do this problem, then at this stage we are done; we have a solution formula which can be justified using the tools from class. QED.

Now, let us return to the original problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for} \quad (x, y) \in \mathbb{H} \\ u_y(x, 0) = g(x). \end{cases}$$

Solution: If we repeat all of the same steps, as above, we end up with an interesting expression for $\hat{u}(\xi, y)$:

$$\hat{u}(\xi, y) = \frac{-g(\xi)}{|\xi|}\exp(-y|\xi|) = g(\xi)\left[\frac{-\exp(-y|\xi|)}{|\xi|}\right].$$

(You should show this, if you try to solve this version of the problem on the HW.)

Attempting to invert (and again invoke the convolution/product rule for $\mathscr{F}$), we run into a *key* problem: The function $\dfrac{-\exp(-y|\xi|)}{|\xi|} \notin L^2(\mathbb{R})$—you can check. The singularity at $\xi = 0$ is non-integrable!

Therefore, we are not guaranteed that this function $\rho(\xi, y) \equiv \dfrac{-e^{-y|\xi|}}{|\xi|}$ can be inverted (this is a famous function, by the way). Writing out the Fourier inversion formula (which may or may not be valid here), we see:

$$R(x,y) \text{``} = \text{''} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \frac{-e^{-y|\xi|}}{|\xi|} d\xi.$$

Unfortunately, this integral cannot be classically evaluated, because the singularity at $\xi = 0$ is non-integrable (classically).

We need a new approach. Let us "think" of our transform $\hat{u}(\xi, y)$ in a different way:

$$\hat{u}(\xi, y) = \hat{g}(\xi) \cdot e^{-y|\xi|} \cdot \frac{-1}{|\xi|}$$

Using induction, we can note that just like for a pointwise product of two functions in the dual domain, we can prove a rule for three functions:

$$\mathscr{F}[f * g * h](\xi) = \hat{f}(\xi)\hat{g}(\xi)\hat{h}(\xi); \quad (f * g * h)(x) = \mathscr{F}^{-1}[\hat{f}\hat{g}\hat{h}](x).$$

With a little work (actually, a lot of work involving distributions), we can invoke all of the constituent transforms of $\hat{u}(\xi, y)$:

$$\begin{aligned} \mathscr{F}_\xi^{-1}[\hat{g}](x) &= g(x) && \text{definition of Fourier inversion} \\ \mathscr{F}_\xi^{-1}[e^{-y|\xi|}](x) &= \frac{1}{\pi}\frac{y}{x^2+y^2} && \text{from Bell's notes, similar calculation to } \mathscr{F}[e^{-c|x|}] \\ \mathscr{F}_\xi^{-1}[-1/|\xi|](x) &= const. + \frac{1}{\pi}\ln|x| && \text{from below} \end{aligned}$$

The last Fourier transform involves a trick: We can show (using distributions) that

$$\mathscr{F}[\ln|x|](\xi) = -2\pi c_e \delta(\xi) - \frac{\pi}{|\xi|},$$

where c_e is a special constant. It is easy to observe that $\mathscr{F}_\xi^{-1}[\delta(\xi)](x) = \dfrac{1}{2\pi}$. From which we deduce:

$$\mathscr{F}_\xi^{-1}[-1/|\xi|](x) = \frac{c_e}{\pi} + \frac{1}{\pi}\ln|x|$$

From this mess, we can infer that the solution has the form:

$$u(x,y) = g(x) *_x \left[\frac{1}{\pi}\frac{y}{x^2+y^2} *_x \left(\frac{1}{\pi}\ln|x| - \frac{c_e}{\pi}\right)\right]$$

In this sense, the *Green's function* $G(x,y)$ is given through this complicated convolution:

$$G(x,y) = \frac{1}{\pi^2}\frac{y}{x^2+y^2} *_x \ln|x|,$$

where we have disregarded the effect of convolving with the constant.

What? This thing keeps going? Yes! There is something cool left to say, but it uses dark magic!

In Bell (as well as in DuZ), the Dirichlet Laplace problem is considered:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x,y) \in \mathbb{H} \\ u(x,0) = f(x). \end{cases}$$

Solving this problem is much easier (hence you have to do it in a previous part), and the steps/solution is readily available. The solution is given by the Green's function: $G(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$. The solution to the Dirichlet problem is thus:

$$u(x, y) = f(x) *_x G(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\tilde{x})}{(x - \tilde{x})^2 + y^2} d\tilde{x},$$

where $\tilde{x}$ is the auxiliary variable associated to the convolution in x.

How is this related to our Neumann problem? Answer: SORCERY! (This is how the problem is solved in DuZ.) Let us reconsider the Neumann problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for} \quad (x, y) \in \mathbb{H} \\ u_y(x, 0) = g(x). \end{cases}$$

But, let's go ahead and differentiate (just) the PDE in y (and not the BC), and use a new variable $w = u_y$. You can verify that by commuting the order of paritals, considering the boundary condition, we obtain a new problem:

$$\begin{cases} w_{xx} + w_{yy} = 0 & \text{for} \quad (x, y) \in \mathbb{H} \\ w(x, 0) = g(x). \end{cases}$$

Welp, this w solves a DIRICHLET problem on the half space, and thus has the solution as above:

$$w(x, y) = g(x) *_x G(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\tilde{x})}{(x - \tilde{x})^2 + y^2} d\tilde{x}.$$

But, $w = u_y$, so we have, in actuality:

$$u_y(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\tilde{x})}{(x - \tilde{x})^2 + y^2} d\tilde{x},$$

and this can be integrated in y! Let's do it:

$$\begin{aligned} u(x, y) &= \int \left[\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\tilde{x})}{(x - \tilde{x})^2 + y^2} d\tilde{x} \right] dy \\ &= \int \frac{1}{\pi} \int_{-\infty}^{\infty} g(\tilde{x}) \frac{y}{(x - \tilde{x})^2 + y^2} d\tilde{x} dy \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} g(\tilde{x}) \left[\int \frac{y}{(x - \tilde{x})^2 + y^2} dy \right] d\tilde{x} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\tilde{x}) \ln[(x - \tilde{x})^2 + y^2] d\tilde{x} + C_{\hat{g}} \end{aligned}$$

where the last integration was a simple substitution in y.

A couple of comments are in order. First, the constant C depends on g above because we have to be able to integrate $\hat{g}$ on $\mathbb{R}$; in other words $\hat{g}$ needs to be in $L^1(\mathbb{R})$, and this is a condition we check. Moreover, this constant was actually a constant with respect to integration in y, and thus $C_{\hat{g}} = C(x)$. But if we plug this solution back into the equation, we will see that $C_{xx} = 0$ which implies that C is linear; in order for it to be bounded (as a function), we must have that $C = const. = C_{\hat{g}}$, as we wrote.

We also note that the Green's function here is clear: $G(x, y) = \frac{1}{2\pi} \ln[x^2 + y^2] + const.$ Thus we have an expression for the solution of the form:

$$u(x, y) = g(x) *_x G(x, y).$$

Now, we have not show it, but if we assume uniqueness of the solution (as we typically obtain, and as does hold here, at least in some functional setting) we have two expressions for the solution of the form: $u(x, y) = g *_x G(x, y)$, where $G(x, y)$ is represented in two different ways. Thus, in a roundabout way, we seem to have proven (via uniqueness) that

$$G(x, y) = \frac{1}{2\pi} \ln[x^2 + y^2] + C_1. = \frac{1}{\pi^2} \frac{y}{x^2 + y^2} *_x \ln|x| + C_2,$$

where C_1 and C_2 indicate that these formulas may differ by a constant. This relation would certainly be worth checking.

Okay, let's just do it! I mean, this thing is already a scroll of unabridged dictionary proportions. Let's compute the convolution in x $\frac{1}{\pi^2} \frac{y}{x^2 + y^2} *_x \ln|x|$.

$$\begin{aligned} \frac{1}{\pi^2} \frac{y}{x^2 + y^2} *_x \ln|x| &= \frac{y}{\pi^2} \int_{\mathbb{R}} \frac{\ln|\tilde{x}|}{(x - \tilde{x})^2 + y^2} d\tilde{x} \\ &= \frac{y}{\pi^2} \left[\frac{\pi}{2y} \ln(x^2 + y^2) \right] \\ &= \frac{1}{2\pi} \ln(x^2 + y^2) \end{aligned}$$

OMG. They are the same. I can't believe it!

By the way, to evaluate the integral above, one must use *contour methods* and the famous *keyhole* contour. It is a highly nontrivial integral—see MATH 410.

See? This thing has an end.