10/23/2025

Heat Equation Activity II Solutions

MATH 404, Fall 2025

1. **Slowing Down Infinite Speed of Propagation**

(a) Consider the convolution solution: $(\phi * S)(x,t) = \int_{-\infty}^{\infty} \phi(y)S(x-y,t)dy$. Explain in a sentence or two what infinite speed of propagation means in terms of this solution.

We note that to compute the (now unique) solution $\phi * S$ at *any* point in space-time, (x,t), we must know the initial data $\phi(y)$ everywhere in space—to compute an integral of this form (which constitutes a weighted average of the initial data and the heat kernel), we have to have point-wise values of ϕ everywhere in $\mathbb{R}$. We think of this as infinite speed because a change (or perturbation) to the initial data ϕ *anywhere* instantanousely changes the solution $u(x,t) = (\phi * S)(x,t)$ *everywhere* in space-time. Hence, instantaneous change $\implies$ infinite speed (of propagation of information).

(b) On the back of this page, consider replacing Fourier's/Darcy's/Ficke's Law for flux $q = -Du_x$ with the relaxation (Maxwell-Cattaneo) Law $\tau q_t + q = -Du_x$, where $\tau > 0$ is a parameter. Differentiate the conservation law $u_t + \partial_x q = 0$ in time, and differentiate the MC law in space. Combine them to eliminate the variable q. What type of equation do you now have? Explain in a sentence or two.

The system that originally gave us the heat equation was algebraic and differential:

$$\begin{cases} u_t + q_x = 0 \\ q = -Du_x, \end{cases} \tag{1}$$

where u is the quantity of interest (operating as a density or concentration) in a conservation law, and q is the flux (defined in terms of u) by a constitutive relation. Putting these together, it is clear we get the HEQ.

We now replace the constitutive law above with $\tau q_t + q = -Du_x$ (the Maxwell-Cattaneo Law). This represents introducing a so-called relaxation time, with a relaxation parameter $\tau > 0$. We then have the system:

$$\begin{cases} u_t + q_x = 0 \\ \tau q_t + q = -Du_x. \end{cases} \tag{2}$$

Taking ∂_t of the first equation and ∂_x of the second, we obtain:

$$\begin{cases} u_{tt} + q_{xt} = 0 \\ \tau q_{xt} + q_x = -Du_{xx}. \end{cases} \tag{3}$$

Using algebra, we obtain:

$$\tau u_{tt} - q_x - Du_{xx} = 0.$$

We then recall that the conservation law says that $q_x = -u_t$, so we can eliminate q to obtain the final PDE in u:

$$u_{tt} + [1/\tau]u_t - [D/\tau]u_{xx} = 0.$$

Recall that this is a damped wave equation. The damping coefficient is $[1/\tau]$ and the wave speed is $c = [D/\tau]^{1/2}$.

(c) Explain how changing the *constitutive law* provided finite speed of propagation, without destroying the *diffusive* nature of the problem. By introducing the relaxation parameter τ in the constitutive law (i.e., Fourier/Darcy/Fick $\mapsto$ Maxwell-Cattaneo), we went from the HEQ to a damped WEQ. The HEQ is diffusive and has the property of infinite speed of propagation. The damped WEQ is also diffusive, but has finite speed of propagation. In this way retain one essential feature of the dynamics, but eliminate one (perhaps) bothersome one.

2. **Motivating Fourier**

(a) Recall the wave-train form: $u(x,t) = Ae^{i(kx-\omega t)}$. Plug in to the HEQ and solve for $\omega(k)$. After doing this, plug back into the wave train form to get an expression for u only in terms of k.

We have solved for the dispersion relation for the HEQ before. We obtain, plugging the wave-train into the HEQ: $-i\omega = D(ik)^2$, which yields $\omega(k) = -iDk^2$. Plugging this into the wave train yields:

$$u(x,t) = A\exp i(kx + iDk^2 t) = Ae^{-Dk^2 t}e^{ikx}.$$

This function is complex-valued, i.e., $\hat{u}(k) \in \mathbb{C}$, and solves the HEQ for each k.

Note that in this construction, the constant A can actually depend on the wave number, k, without modifying anything. Thus we have a general expression, which we will call $\tilde{u}(k,t) = A(k)e^{-Dk^2 t}e^{ikx}$. What can we do with it?

Well, if it solves the HEQ, we'll need it to satisfy an initial condition. We seem to have $\hat{u}(k,0) = A(k)e^{ikx}$. That isn't too helpful...

(b) Now, since we have a function for *each* k, we can *superpose* all of them to obtain another general solution for the HEQ. This produce an integral of the form

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{ikx}e^{-Dk^2 t}dk.$$

We can check to see (differentiating through integrals) that this solves the HEQ. So what about that initial condition (for the Cauchy problem)? This begs the question: When can we choose $A(k)$ in such a way that

$$u(x,0) = \phi(x) \quad \text{and} \quad u(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk?$$

This question is a motivation for the Fourier transform.

Namely, given $\phi(x)$ find $A(k)$ so that

$$\phi(x) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk.$$