10/2025

Heat Equation Activity: Qualitative Properties and Other Fun!

MATH 404, Fall 2025

1. **Energies.** Define the *thermal energy* of a solution u at time t to be: $E(t) = \frac{1}{2}\int_{-\infty}^{\infty} [u(x,t)]^2 dx.$

   (a) Do either of the following: (i) Multiply the heat equation by $u(x,t)$ and then integrate for all $x \in \mathbb{R}$; then integrate by parts in space in order to obtain an *energy relation* (using the expression for $E(t)$ above); **OR** (ii) Compute d/dt of $E(t)$ and simplify using integration by parts and the heat equation itself.

   Multiplying and integrating give us

$$\int_{-\infty}^{\infty} u_t(\xi,t)u(\xi,t)d\xi - D\int_{-\infty}^{\infty} u_{xx}(\xi,t)u(\xi,t)d\xi = 0.$$

   Using the relation $\frac{d}{dt}[f^2] = 2f'(t)f(t)$ and integrating by parts in x in the second term, we have

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} [u(\xi,t)]^2 d\xi + D\int_{-\infty}^{\infty} [u_x(\xi,t)]^2 d\xi - D\big[u_x(\xi,t)u(\xi,t)\big]_{x\to-\infty}^{x\to\infty} = 0$$

   Since we have assumed that the solution $u(x,t)$ decays to zero as $|x| \to \infty$, the boundary term goes away, yielding

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} [u(\xi,t)]^2 d\xi + D\int_{-\infty}^{\infty} [u_x(\xi,t)]^2 d\xi = 0.$$

   Then, we can use the FTC to rewrite:

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} [u(\xi,t)]^2 d\xi = -D\int_{-\infty}^{\infty} [u_x(\xi,t)]^2 d\xi \quad \Longleftrightarrow \quad E(t) = E(0) - D\int_0^t \int_{-\infty}^{\infty} [u_x]^2 d\xi d\tau.$$

   The first equality above can also be rewritten, using the definition of the energy $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}[u(x,t)]^2 dx$ as

$$\frac{d}{dt}E(t) = -D||u_x||^2_{L^2(\mathbb{R})}.$$

   (b) Explain, from your previous answer, why *energies decay* for solutions to the heat equation.

   From the first description above, we see that the time rate of change of $E(t)$ is negative, indicating decay. From the second description, it is clear that since $[u_x]^2$ is a nonnegative function, its integral is also nonnegative and being subtracted from $E(0)$, so $E(t) \leq E(0)$. In fact, for any interval of time $[s,t]$, we will have $E(t) \leq E(s)$, which is another way of saying that $E(t)$ is a decreasing function.

2. **Superposition and Differences**.

   (a) Suppose that $v(x,t)$ is a solution to equation (**??**) with initial data $v(x,0) = \phi_1(x)$ and $w(x,t)$ is another solution with $w(x,0) = \phi_2(x)$.

   What Cauchy problem does $z = v - w$ solve?

   By the principle of superposition, any linear combination of solutions (to a linear, homogeneous PDE) is also a solution. Hence $z = v - w$ solves the HEQ. We can actually just show this too, since:

$$z_t = [v-w]_t = v_t - w_t = Dv_{xx} - Dw_{xx} = D\partial_x^2[v-w] = Dz_{xx}.$$

   It is clear that $z(x,0^+) = v(x,0^+) - w(x,0^+) = \phi_1(x) - \phi_2(x)$.

   So the cauchy problem that z solves is just:

$$\begin{cases} u_t - Du_{xx} = 0, & x \in \mathbb{R},\ t > 0 \\ \lim\limits_{t\searrow 0}\ u(x,t) = \phi_1(x) - \phi_2(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

(b) Since z solves the heat equation, it satisfies the energy relation from the previous part. Write this out for z in terms of $L^2(\mathbb{R})$ norms. Use the notation $E_z(t) = \frac{1}{2}\int_{-\infty}^{\infty}[z(\xi,t)]^2 d\xi$. Recall:

$$\int_{-\infty}^{\infty}[z(x,t)]^2 dx = \int_{-\infty}^{\infty}[v(x,t)-w(x,t)]^2 dx = ||z||^2_{L^2(\mathbb{R})} = ||v-w||^2_{L^2(\mathbb{R})}.$$

Since *any* solution to the HEQ satisfies the energy relation $E(t) \leq E(0)$, and z is a solution, we know that, in particular:

$$E_z(t) \leq E_z(0) \quad \Longleftrightarrow \quad \frac{1}{2}||v(x,t)-w(x,t)||^2_{L^2(\mathbb{R})} \leq \frac{1}{2}||v(x,0)-w(x,0)||^2_{L^2(\mathbb{R})}.$$

This can be re-written once more as

$$||v-w||^2 \leq ||\phi_1 - \phi_2||^2.$$

3. **Uniqueness and Continuous Dependence**

(a) Using the energy relation for $E_z(t)$ from the previous part, explain (in a couple sentences) why the HEQ has the *continuous dependence property* in the sense of $L^2(\mathbb{R})$.

By the above inequality,

$$||v-w||^2 \leq ||\phi_1 - \phi_2||^2,$$

we see that we can make the solutions v and w as close as we want (in the sense that we can make the norm $||v-w||_{L^2(\mathbb{R})}$ small) by choosing the initial data ϕ_1 and ϕ_2 close (in the sense that $||\phi_1 - \phi_2||_{L^2(\mathbb{R})}$ is small). Thus, the solutions can be made "close together" at time t by choosing the data ϕ_1 and ϕ_2 to be sufficiently "close together."

(b) Suppose that v and w are two solutions that have the same initial conditions (i.e., $\phi_1(x) = \phi_2(x)$). What does the energy relation for $E_z(t)$ say in this case?

In this case, we would have $E_z(0) = (.5)||\phi_1 - \phi_2||^2 = 0$. Then by the energy inequality, we would have:

$$E_z(t) \leq E_z(0) = 0 \quad \Longrightarrow \quad 0 = E_z(t) = \frac{1}{2}||v-w||^2.$$

From this we see that $v(x,t) - w(x,t) = 0$ as functions in $L^2(\mathbb{R})$ for each $t \geq 0$. But this means that $v(x,t) = w(x,t)$, and hence the solution to the heat equation Cauchy problem must be unique.

4. **Well-posedness**

Explain in one or two sentences why the HEQ Cauchy problem is *well-posed*.

We have established both Continuous Dependence (on the data) and Uniqueness for the HEQ Cauchy problem. We know that the problem *has* a solution, since the convolution solution exists and can solve the Cauchy problem for arbitrary data ϕ. Hence we have existence, uniqueness, and continuous dependence for the Cauchy problem, making it *well-posed*.

5. **Bonus Exploration: Maxwell-Cattaneo**

We replace Fourier's/Darcy's/Ficke's Law for flux $q = -Du_x$ with the relaxation (Maxwell-Cattaneo) Law $\tau q_t + q = -Du_x$, where $\tau > 0$ is a parameter. Differentiate the conservation law $u_t + \partial_x q = 0$ in time, and differentiate the MC law in space. Combine them to eliminate the variable q. What type of equation do you now have? Explain in a sentence or two.

The system that originally gave us the heat equation was algebraic and differential:

$$\begin{cases} u_t + q_x = 0 \\ q = -Du_x, \end{cases} \tag{2}$$

where u is the quantity of interest (operating as a density or concentration) in a conservation law, and q is the flux (defined in terms of u) by a constitutive relation. Putting these together, it is clear we get the HEQ.

We now replace the constitutive law above with $\tau q_t + q = -Du_x$ (the Maxwell-Cattaneo Law). This represents introducing a so-called relaxation time, with a relaxation parameter $\tau > 0$. We then have the system:

$$\begin{cases} u_t + q_x = 0 \\ \tau q_t + q = -Du_x. \end{cases} \tag{3}$$

Taking ∂_t of the first equation and ∂_x of the second, we obtain:

$$\begin{cases} u_{tt} + q_{xt} = 0 \\ \tau q_{xt} + q_x = -Du_{xx}. \end{cases} \tag{4}$$

Using algebra, we obtain:

$$\tau u_{tt} - q_x - Du_{xx} = 0.$$

We then recall that the conservation law says that $q_x = -u_t$, so we can eliminate q to obtain the final PDE in u:

$$u_{tt} + [1/\tau]u_t - [D/\tau]u_{xx} = 0.$$

Recall that this is a damped wave equation. The damping coefficient is $[1/\tau]$ and the wave speed is $c = [D/\tau]^{1/2}$.