10/20/2025

Heat Equation Activity II

MATH 404, Fall 2025

This activity is worth 5 points total (in-class points = quiz/activity points). It is due at the beginning of class on Wednesday.

The homogeneous heat/diffusion equation is given by $u_t - Du_{xx} = 0$, where $u(x,t)$ is the solution variable giving the "quantity" (temperature, energy, concentration) of a substance at x and time t. Taken with initial heat distribution, $\phi(x)$, we have the homogeneous **heat equation Cauchy problem**:

$$\begin{cases} u_t - Du_{xx} = 0, & x \in \mathbb{R},\ t > 0 \\ \lim\limits_{t \searrow 0} u(x,t) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

The unique solution (supposing that $\phi \in L^2(\mathbb{R})$) is given by the convolution solution:

$$u(x,t) = S(x,t) * \phi(x) = \int_{-\infty}^{\infty} S(y,t)\phi(x-y)dy = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy. \tag{2}$$

1. **Slowing Down Infinite Speed of Propagation**

   (a) Consider the convolution solution above. Explain in a sentence or two what infinite speed of propagation means in terms of this solution. (Think about modifying some value of $\phi(x)$ and its effect on the solution, and compare against the same effect on the wave equation.)

   (b) When we replaced Fourier's/Darcy's/Ficke's Law for flux $q = -Du_x$ with the relaxation (Maxwell-Cattaneo) Law $\tau q_t + q = -Du_x$ ($\tau > 0$ is a parameter), we obtained the equation:

   $$u_{tt} + [1/\tau]u_t - [D/\tau]u_{xx} = 0.$$

   What type of equation do you now have? Explain in a sentence or two.

(c) Explain how changing the *constitutive law* provided finite speed of propagation, without destroying the *diffusive* nature of the problem.

2. **Motivating Fourier**

(a) Recall the wave-train form: $u(x,t) = Ae^{i(kx-\omega t)}$. Plug in to the HEQ and solve for $k(\omega)$. After doing this, plug back into the wave train form to get an expression for u only in terms of k.

(b) Now allow $A = A(k)$. *Superpose* all of the solutions $u(k)$ in an integral of the form $\int_{-\infty}^{\infty} u(k)dk$. Stare at the formula for at least three-five minutes.