Homog.

Example 2 Neumann Heat EQ on $\mathbb{R}_+$ ①

$$
(*)\begin{cases} u_t - D u_{xx} = 0 & x>0,\ t>0 \\ u(x,0^+) = \varphi(x) & x>0 \\ u_x(0,t) = 0 & t \geq 0 \end{cases}
$$

$\varphi(0) \overset{?}{=} 0$ compatibility

Consider $\varphi^e(x) = \begin{cases} \varphi(x) & x>0 \\ \varphi(-x) & x<0 \end{cases}$

Then create the auxiliary problem:

$w(x,t)$ for $x \in \mathbb{R},\ t \geq 0$:

$$
(**)\begin{cases} w_t - D w_{xx} = 0 & x \in \mathbb{R},\ t>0 \\ w(x,0) = \varphi^e(x) & x \in \mathbb{R} \end{cases}
$$

The Cauchy problem above has unique

sol^n: $w(x,t) = \varphi^e(x) * S(x,t)$

Let's try to restrict to $x>0$ and simplify. Compute: $\varphi^e(x) * S(x,t)\big|_{x>0}$

For $x > 0$:

$$\varphi^e(x) * S(x,t) = \int_{-\infty}^{\infty} \varphi^e(y) S(x-y,t)\,dy$$

$$= \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \varphi^e(y) S(x-y,t)\,dy$$

$$= \int_0^{\infty} \varphi(y) S(x-y,t)\,dy$$

let $z = -y$
$dz = -dy$ $\rightarrow$ $$+ \int_{-\infty}^{0} \varphi(-y) S(x-y,t)\,dy$$

$$= \int_0^{\infty} \varphi(y) S(x-y,t)\,dy$$

$$- \int_{\infty}^{0} \varphi(z) S(x+z,t)\,dz$$

z is just "dummy" $$= \int_0^{\infty} \varphi(y) \left[S(x-y,t) + S(x+y,t)\right] dy$$

Lets let

$$u(x,t) = w(x,t)\Big|_{x>0}$$ (the restriction of $\varphi^e * S$ to $x>0$)

Does it solve $*$?

u solves HEQ on $\mathbb{R}_+$ since

$u = w|_{x>0}$ and w solves HEQ on $\mathbb{R}$.

Let's check the BC: $u_x(0,t)$

$$u_x(x,t) = \int_0^\infty \varphi(y)\left[S_x(x-y,t) + S_x(x+y,t)\right]dy$$

$$= \int_0^\infty \varphi(y)\left[\frac{-(x-y)}{2Dt} S(x-y,t) - \frac{(x+y)}{2Dt} S(x+y,t)\right]$$

so

$$u_x(0,t) = \frac{1}{2Dt}\int_0^\infty \varphi(y)\left[y\,S(-y,t) - y\,S(y,t)\right]dy$$

$$= 0 \checkmark \quad \text{since } S(-y,t) = S(y,t)$$

Now the IC's:

$$u(x,0) = \lim_{t \downarrow 0} \int_0^\infty \varphi(y) S(x-y,t)\,dy$$
$$+ \lim_{t \downarrow 0} \int_{-\infty}^0 \varphi(-y) S(x-y,t)\,dy$$

let $z = \frac{x-y}{2\sqrt{Dt}}$ so $dz = -\frac{dy}{2\sqrt{Dt}}$

and $y = x - (2\sqrt{Dt})z$

So

$$u(x,0^+) = \lim_{t \downarrow 0} \frac{-1}{\sqrt{\pi}} \int_{x/2\sqrt{\kappa t}}^{-\infty} \varphi(x - 2\sqrt{\kappa t}\, z)\, e^{-z^2}\, dz$$

$$+ \lim_{t \downarrow 0} \frac{-1}{\sqrt{\pi}} \int_{\infty}^{x/2\sqrt{\kappa t}} \varphi(2\sqrt{\kappa t}\, z - x)\, e^{-z^2}\, dz$$

Since $x > 0$

$$= -\frac{1}{\sqrt{\pi}} \int_{\infty}^{-\infty} \varphi(x)\, e^{-z^2}\, dz$$

$$\cancel{- \frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} \varphi(-x)\, e^{-z^2}\, dz} \to 0$$

$$= \varphi(x) \checkmark$$

Let's talk about uniqueness.

If $u^1(x,t)$ & $u^2(x,t)$ solve an $\mathbb{R}_+$ IBVP, then their (appropriate) extensions $w^1(x,t)$ & $w^2(x,t)$ satisfy a well-posed Cauchy problem on $\mathbb{R}$. Hence, $u^1 = u^2$.

[odd extensions of homog. Neumann data even extensions of homog. Dirichlet data are not candidates for sol'ns. Why?]