ex 1 Homog Dirichlet Wave on $\mathbb{R}_+$.

$$\# \begin{cases} u_{tt} - c^2 u_{xx} = 0 & x > 0,\ t > 0 \\ u(0,t) = 0 & t \geq 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) & x > 0 \end{cases}$$

So Form $\varphi^o(x)$ & $\psi^o(x)$.

Consider the problem:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ w(x,0) = \varphi^o(x) & x \in \mathbb{R} \\ w_t(x,0) = \psi^o(x) \end{cases}$$

Has sol$^{\underline{n}}$: d'Alembert

$$w(x,t) = \frac{1}{2}\{\varphi^o(x+ct) + \varphi^o(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi^o(s)\,ds$$

Let's try to simplify a bit by cases: $\boxed{x \geq 0}$

$x > ct$	$x+ct,\ x-ct > 0$
$x < ct$	$ct - x > 0 \Rightarrow x - ct < 0$

Let's restrict $w(x,t)$ to $x>0$.

when $x>ct$:

$$\varphi^{o}(x+ct) = \varphi(x+ct), \quad \varphi^{o}(x-ct) = \varphi(x-ct)$$

same for ψ^{o}/ψ

when $x<ct$, $x+ct>0$ ✓, $\boxed{\underline{x-ct<0}}$

$$\varphi^{o}(x+ct) = \varphi(x+ct), \quad \varphi^{o}(x-ct) = \varphi^{o}(-(ct-x))$$
$$= -\varphi(ct-x)$$

$\underline{\text{so}}$ $w(x,t)\big|_{x>0}$ is:

$$u(x,t) = \begin{cases} \frac{1}{2}\{\varphi(x+ct)+\varphi(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds \\ \quad \underline{\text{when}}\ x>\underline{ct} \\ \frac{1}{2}\{\varphi(x+ct)-\varphi(ct-x)\} + \frac{1}{2c}\int_0^{x+ct}\psi(s)\,ds \\ \quad \underline{\text{when}\ x<ct} \qquad - \frac{1}{2c}\int_0^{ct-x}\psi(s)\,ds \end{cases}$$

$$\int_{x-ct}^{0}\psi^{o}\,ds \underset{(\text{oddness})}{=} -\int_0^{(ct-x)}\psi^{o}\,ds \underset{\left(\text{def}^{n}\text{ of }\psi^{o}\right)}{=} (-1)\int_0^{ct-x}\psi\,ds$$

So $w(x,t)\big|_{x>0} = u(x,t)$ solves the half space problem ✱. And what is

$$u(0,t) = \begin{cases} \varphi(0) & x=t=0 \\ 0 & x=0,\ t>0 \end{cases}$$

So If $\varphi(0)=0$ (compatibility condition) then $u(x,t)\big|_{x=0} = 0$. Thus we have solved the BVP. Let's check ICs:

$u(x,0) = \varphi(x)$, since $x > ct$

$u_t(x,0) = \psi(x)$, since $x > ct$

at $x=t=0$,

$$u(0,0) = \begin{cases} \varphi(0) & (x-ct)^+ \\ 0 & (x-ct)^- \end{cases}$$

Again, if $\varphi(0)=0$, the IVs satisfied.

So $u(x,t) = w(x,t)\big|_{x\geq 0}$ is the sol^n to the IBVP!