November 16, 2025

Homework 7: The Boundary and Separation

MATH 404, Fall 2025

This assignment is due: 12/3 (in class).

This assignment is about boundary value problems and separation of variables.

1. (Bell 15, Strauss 4.3) **The Robin EVP**. Consider the general Robin EVP.

$$\begin{cases} s_{xx}(x) + \lambda s(x) = 0 \\ s_x(0,t) - a_0 s(0,t) = 0 \\ s_x(L,t) + a_L s(L,t) = 0. \end{cases}$$

   The values a_0, a_L (their signs and magnitudes) have immense bearing on the eigenvalues. This problem is quite involved—see the reading. We will focus on simpler cases here.

   (a) When $a_0, a_L > 0$, the boundary conditions are called *radiation conditons*. When $a_0, a_L < 0$ they are called *absorption conditions*. Explain this convention in a couple of sentences.

   (b) Consider the simplified problem with $L = 1$, and $a_L > 0$:

$$s(0,t) = 0; \quad s_x(1,t) + a_L s(1,t) = 0.$$

   Show that $\lambda > 0$ for any solution.

   (c) Under the assumptions of the prevsious part, derive the transcendental equation which characterizes the values of λ_n. Show your work, and provide the approximate values of λ_i for $i = 1, 2, 3$.

2. **Energy Methods**. We will use the energy relation to show uniqueness. Consider the following IBVP (of Dirichlet type) for the heat equation for $x \in [0,1]$:

$$\begin{cases} u_t = u_{xx} & x \in (0,1),\ t > 0, \\ u(x,0) = \phi(x) & x \in [0,1], \\ u(0,t) = f(t); \quad u(1,t) = g(t) & t \geq 0. \end{cases}$$

   Let u_1 be a solution with data $\phi_1(x), f_1(t), g_1(t)$ and u_2 be a solution with data $\phi_2, f_2(t), g_2(t)$.

   (a) Derive an energy relation for this problem using $E(t) = \frac{1}{2}||u||^2_{L^2(0,1)}$.

   (b) Use this energy relation to deduce *uniqueness* of solutions in the sense of $L^2(0,1)$.

   (c) For $f_i = g_i \equiv 0$, explain how the energy relation also gives *continuous dependence* on the initial data in the sense of $L^2(0,1)$.

   (d) (Strauss 2.3 #8) Now consider the heat equation above, but with the Robin conditions replacing the Dirichlet conditions:

$$\begin{cases} u_x(0,t) - a_0 u(0,t) = 0 \\ u_x(L,t) + a_L u(L,t) = 0. \end{cases}$$

   Find the energy relation for this problem and then comment on why, if $a_0, a_L > 0$, we say that the boundary conditions are *dissipative*.

3. (Strauss 4.2, Bell 14) **Separation of Variables for Mixed BCs**.

   (a) Complete the full separation of variables argument (as we did in class) for the mixed BVP heat equation:

   $$\begin{cases} u_t = Du_{xx} \\ u(0,t) = u_x(L,t) = 0. \end{cases}$$

   You should separate, solve the eigenvalue problem, and superpose. Note that we do not include initial conditions, and thus you should give your answer in the most general sense (incorporating the most general initial condition possible).

   (b) What happens to the solution as $t \to \infty$?

   (c) Without repeating all of the previous steps, what changes for your solution if you consider the wave equation $u_{tt} = c^2 u_{xx}$ (with the same boundary conditions) as above? Give your complete solution *being careful not to do any unnecessary work.*

4. (Bell 14, #5) **Wave with Damping**. Consider the homogeneous Dirichlet-Neumann mixed wave equation with damping on $x \in [0,1]$:

   $$\begin{cases} u_{tt} - c^2 u_{xx} + du_t = 0 \\ u(x,0) = \phi(x); \quad u_t(x,0) = 0 \\ u(0,t) = u_x(1,t) = 0 \end{cases}$$

   Use separation of variables to obtain the series solution when $d < \pi c$. You may refer to your work in previous problems, and, again, do not do unnecessary work.

5. (Strauss, pp.91–92) **Additional Separations**. What are the eigenvalues and eigenfunctions associated to the following problems on $x \in [0, L]$? Show your work, but you may be concise—note that you are not being asked to solve the entire problem.

   (a) $u_t = \dfrac{i\hbar}{2m} u_{xx}$ with $u(0,t) = u(L,t) = 0$? ($\hbar$ and m are positive constants; $i = \sqrt{-1}$.)

   (b) $u_{tt} = c^2 u_{xx}$ with $u_x(0,t) = u(L,t) = 0$?

   (c) $u_{tt} + E\partial_x^4 u = 0$ with $u(0,t) = u_{xx}(0,t) = 0,\ u(L,t) = u_{xx}(L,t) = 0$? ($E > 0$.)

6. (Strauss 4.2 #1) **Eigenmusic**. Consider the Dirichlet wave equation with homogeneous boundary conditions as a model for a violin string.

   (a) Use the series solution to explain (in a sentence or two) why "clamping" the string exactly at its midpoint raises the pitch by one octave.

   (b) In a similar way, explain why the pitch increases as the string is tightened.

7. (Bell 14 # 9) **Inverse Problem.** Let $a(t)$ be a continuous, positive function on $[0, \infty)$. Consider the problem

   $$\begin{cases} u_t - a(t)u_{xx} = 0 \\ u(x,0) = \sin(\pi x); \quad u_t(x,0) = 0 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

   (a) Solve for $u(x,t)$, assuming that the function $a(t)$ is known.

   (b) Solve the inverse problem: assume that $a(t)$ is unknown, but suppose that we know $u(.5,t) = g(t)$ for all $t > 0$, and that $g(t)$ is positive, continuously differentiable, and $g'(t) < 0$ for all $t > 0$ with $g(0) = 1$. Solve for $a(t)$.

8. **Steady State and Dirichlet Lift**.

   (a) Consider the *steady state* problem $u_{xx} = 0$ on $x \in [0, L]$. Note that this is the steady state problem for both the heat and wave equation. Assume the solution $u(x)$ is smooth. What can you deduce about it (using Calc. I)? Explain how the boundary conditions at $x = 0$ and $x = L$ are involved here.

   (b) Now consider the inhomogeneous Dirichlet problem for the heat equation with $x \in [0, L]$:

   $$\begin{cases} u_t - Du_{xx} = 0, \quad x \in (0, L) \\ u(x, 0) = \phi(x) \\ u(0, t) = g_1(t); \quad u(L, t) = g_2(t). \end{cases}$$

   We will transform this problem into one with homogeneous boundary conditions.

   i. Assuming that $g_i(t)$ are given for all t, compute a linear function $\mathcal{L}(x, t)$ which connects $(0, g_1(t))$ to $(L, g_2(t))$ for any value of t.

   ii. Consider a new variable $w(x, t) = u(x, t) - \mathcal{L}(x, t)$. Assuming $u(x, t)$ is a solution, plug w this into the PDE. What PDE does w solve?

   iii. You achieved homogenous boundary conditions; what was the *cost* of doing this?

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9. (Bonus) **Steady State, II**.

   (a) Suppose that the heat flow in a metal rod $x \in [0, 1]$ is governed by the following equations

   $$\begin{cases} u_t = Du_{xx} + 1 \\ u(0, t) = 0, \ u(1, t) = 1. \end{cases}$$

   What will be the steady state temperature after a long time? Explain why it does not matter that no $u(x, 0) = \phi(x)$ is given.

   (b) Consider, instead, that the heat flow in a metal rod $x \in [0, 1]$ is governed by

   $$\begin{cases} u_t = Du_{xx} - hu \\ u(0, t) = u(1, t) = 1, \end{cases}$$

   with $h > 0$. Find the steady state solution and describe what the heat flow is doing in the dynamics.

10. (Bonus) **Solve the Clamped Beam.** Consider the beam equation describing the out-of-plane displacements (as with the wave equation) of an elastic beam defined on $[0, L]$ taken with clamped boundary conditions:

    $$u_{tt} + E\partial_x^4 u = 0, \quad u(x, 0) = \phi(x), \ u_t(x, 0) = \psi(x); \quad u(0, t) = u_x(0, t) = u(L, t) = u_x(L, t) = 0.$$

    The constant E is an elastic constant corresponding to the beam material and its geometry. Give the serious solution to the above, using the eigenvalues and eigenfunctions obtained in an earlier problem—do not repeat work!

11. (Bonus) Consider the following IBVP (of Neumann type) for the wave equation for $x \in [0, 1]$:

$$\begin{cases} u_{tt} = u_{xx} & x \in (0,1),\ t > 0, \\ u(x,0) = \phi(x);\ u_t(x,0) = \psi(x) & x \in [0,1], \\ u_x(0,t) = 0;\quad u_x(1,t) = 0 & t \geq 0. \end{cases}$$

Let u_1 be a solution with data $\phi_1(x), \psi_1(x)$ and u_2 be a solution with data $\phi_2(x), \psi_2(x)$.

(a) Let $z = u_1 - u_2$. Using the the IBVP that z solves, derive the energy relation for

$$E_z(t) = \frac{1}{2}\left\{||z_t||^2_{L^2(0,1)} + ||z_x||^2_{L^2(0,1)}\right\}.$$

(b) Use this energy relation to deduce: *conservation of energy*, *uniqueness* of solutions in the sense of $L^2(0,1)$, and *continuous dependence* in the senes of $L^2(0,1)$. Write a clear sentence for each of these three points.

12. (Bonus) Consider the more general Robin boundary conditions

$$\begin{cases} s_x(0,t) - a_0 s(0,t) = 0 \\ s_x(L,t) + a_L s(L,t) = 0. \end{cases}$$

Assume that $a_0, a_L > 0$. In this case, again, we must have $\lambda > 0$. Derive the transcendental equation (showing work) that characterizes the eigenvalues λ_n.