10/26/2025

Homework 6: Fourier, Laplace, and Semi-infinite Domains

MATH 404, Fall 2025

This assignment is due: W, 11/12, in class. This assignment is about the Fourier transform, half-space problems, and the Laplace transform. It is highly advisable to check the references!

1. Although the Heaviside function $H(x)$ is not in an $L^2(\mathbb{R})$ function, we will give meaning to its Fourier transform here, using the $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ function and some facts:

$$\mathscr{F}[\delta(x)] = 1; \quad \mathscr{F}[1] = \frac{1}{2\pi}\delta(x); \quad \mathscr{F}^{-1}[1] = \delta(x)$$

$$H(x) = \frac{1}{2}[1 + \operatorname{sgn}(x)].$$

   (a) Let $s(x) = \lim_{a\to 0} \left[e^{-ax}H(x) - e^{ax}H(-x)\right]$. Make the case that $s(x) = \operatorname{sgn}(x)$. (It may help to draw a graph or two.)

   (b) Compute $\mathscr{F}[s(x)]$. (You may interchange the integral and the limit in a.)

   (c) Compute $\mathscr{F}[H(x)]$ using linearity of the Fourier transform, and the identity $H(x) = \frac{1}{2}\left[1 + s(x)\right]$.

2. Show the convolution property of the Fourier transform:

$$\mathscr{F}[(f * g)(x)] = \hat{f}(\xi)\hat{g}(\xi), \quad \text{where} \quad (f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy, \quad \forall \; f, g \in L^2(\mathbb{R}).$$

3. (Bell p. 4–5; DuZ p.221) Solve the Cauchy problem for the homogeneous wave equation:

$$u_{tt} - c^2 u_{xx} = 0; \quad u(x,0) = \phi(x), \; u_t(x,0) = \psi(x)$$

   using the Fourier transform, its properties, and its inversion. Show all details, clearly.

4. (Bell p. 5–6; DuZ pp.207–209) Consider Laplace's equation on the half space

$$\mathbb{H} \equiv \{(x,y) \;:\; y > 0\}$$

   for the solution variable $u(x,y)$ with Dirichlet boundary data:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for} \quad (x,y) \in \mathbb{H} \\ u(x,0) = f(x). \end{cases}$$

   (a) Solve the BVP above by taking the Fourier transform in x and enforcing the extra condition that $u(x,y)$ and the transform $\hat{u}(\xi,y)$ stay bounded.

   (b) Consider the Dirichlet boundary data replaced by Neumann data:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 & \text{for} \quad (x,y) \in \mathbb{H} \\ u_y(x,0) = g(x). \end{cases}$$

   Solve the BVP. This can be done by Fourier, but there is a slicker way–see the DuZ reference.

5. (DuZ pp.223–225; Bell 11.1, #3) Consider the (dispersive) Klein-Gordon equation:

$$u_{tt} - c^2 u_{xx} + m^2 u = 0.$$

(a) Compute the Fourier transform of the PDE and solve the resulting ODE to obtain $\hat{u}(\xi, t)$.

(b) DO NOT TRY TO INVERT THIS. Instead, recall the exponential shift property of the Fourier transform, and try to explain the *dispersion* phenomenon associated with the solution by only looking at the form of $\hat{u}(\xi, t)$. (It will benefit you to look in DuZ.)

6. Consider the homogeneous Neumann wave equation IBVP on the half space $x > 0$:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } \ x > 0 \\ u_x(0,t) = 0 & \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) & \text{for } \ x > 0. \end{cases}$$

(a) Label $\phi_e(x)$ and $\psi_e(x)$ the *even extensions* of the initial data and solve:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } \ x \in \mathbb{R} \\ u(x,0) = \phi_e(x), \quad u_t(x,0) = \psi_e(x) & \text{for } \ x \in \mathbb{R}. \end{cases}$$

(b) Show that the solution to the extended problem $u_e(x,t)$ satisfies the boundary condition at $x = 0$.

(c) Deduce the solution to the original IBVP.

7. Repeat the steps from the previous question for the *homogeneous Dirichlet* Heat Cauchy problem. You do not need to show details, but give the appropriate solution.

8. (Bell p. 3; DuZ pp.234–236) Solve the following IBVP problem for the heat equation using the Laplace transform, showing all details:

$$\begin{cases} u_t - D u_{xx} = 0 \\ u_x(0,t) = g(t) \\ u(x,0) = 0. \end{cases}$$

9. Repeat the appropriate analogous steps in problem 3.) for the inhomogeneous Dirichlet wave equation IBVP on $x \in [0, \infty)$:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{for } \ x > 0 \\ u(0,t) = g(t) & \text{for } \ t \geq 0 \\ u(x,0) = 0; \ u_t(x,0) = 0 & \text{for } \ x > 0. \end{cases}$$

10. Consider the general inhomogeneous IBVP for the heat equation for $x > 0$:

$$\begin{cases} u_t - D u_{xx} = F(x,t) & \text{for } \ x > 0 \\ u(0,t) = g(t) & \\ u(x,0) = f(x). & \end{cases}$$

Break this problem into to three subproblems, with a mind toward the *principle of superposition.* Explain, in a few sentences, how you would solve each of the subproblems. Then, finally, explain how you would solve the full problem above.