

Homework 5: Heat and Diffusion**MATH 404, Fall 2024****Due: ~F, 10/17.**

This assignment is about the heat equation on $\mathbb{R}$ (and associated topics).

1. (Bell 2.4, #6) **Gaussian Integral.** Prove that $\int_{-\infty}^{\infty} e^{-r^2} dr = \sqrt{\pi}$.
(There are many ways to do this, but the easiest involves a change to polar coordinates. This cannot be done straightforwardly using the FTC Part 2.)

2. (Strauss 2.4; Bell, 8 p.1) **Fundamental Properties of the Heat Equation and Kernel.**

- (a) Show that if u solves the heat equation, $u_t = Du_{xx}$, then for any $\alpha > 0$, $u(\alpha x, \alpha^2 t)$ is also a solution.
- (b) Consider a solution u to the heat equation $u_t = Du_{xx}$.
 - i. Show that, for any $y \in \mathbb{R}$, $u(x - y, t)$ is also a solution.
 - ii. Explain, in a couple sentences, why

$$\int_{-\infty}^{\infty} f(y)u(x - y, t)dy$$

is also a solution (supposing that f is a nice—say bounded and piece-wise continuous—function).

- iii. Show that u_x, u_t and u_{xx} are all *also* solutions to the heat equation. Convince yourself that *any* derivative of the solution u is also a solution to the heat equation.
- (c) Consider the standard heat kernel $S(x, t) = \frac{1}{2\sqrt{\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right)$.
 - i. Show that $\int_{-\infty}^{\infty} S(x - y, t)dx = 1$ for all $y \in \mathbb{R}$ and all $t > 0$.
(Don't work too hard; use #1.)
 - ii. Verify directly that $S(x, t)$ solves $u_t = Du_{xx}$ for all $t > 0$.

3. (Strauss 2.4, #8, p. 50) **Approximation of the Identity.** Let $S(x, t)$ be the heat kernel.

- (a) Sketch a sequence of pictures of $S(x, t_0)$ for fixed t_0 values as $t_0 \rightarrow 0$
(say for $t_0 = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}$) illustrating the graph of $S(x, t_0)$.
- (b) What does $S(x, t)$ converge to as an object? In what sense? (Think, informally, of the graph of $S(x, 0)$.)

4. Transformations and Reductions.

- (a) (Bell 8, #3) Show that the advection-diffusion-decay equation

$$u_t = Du_{xx} - cu_x - \lambda u$$

can be transformed into a pure diffusion/heat equation using a familiar trick of letting $u(x, t) = w(x, t) \exp(\alpha x - \beta t)$, for convenient values of α and β .

- (b) Show that the equation $u_t = D(t)u_{xx}$ can be transformed into the diffusion equation by changing the time variable t to $\tau = \int_0^t D(s)ds$.

5. **Solving the Heat Equation.** Consider the Cauchy problem on $(0, T) \times \mathbb{R}$: $u_t = Du_{xx}$, $u(x, 0) = \phi(x)$. Find the solution with the following initial data ϕ , and *simplify as much as possible*:

- (a) (Bell 8.2, #1) $\phi(x) = 2x$.
- (b) $\phi(x) = 1 + 3H(x)$. Recall that $H(x)$ is the standard Heaviside function. (Use linearity.)
- (c) $\phi(x) = \cos(bx)$. (For this problem, use the Ansatz that $u(x, t) = e^{at} \cos(bx)$ —you then only have to find the correct a and b .)
- (d) (Strauss 2.4) $\phi(x) = e^{-x}H(x)$. (Write your solution in terms of the erf function; you may need to *complete the square* in the exponent.)

6. **Waves and Speed of Propagation.** Recall that a traveling wave solution $u(x, t) = f(x + \sigma t)$ to the wave equation $u_{tt} - c^2 u_{xx} = 0$ must have $\sigma = \pm c$. Also recall that the concepts of *domain of dependence/influence* had a fundamental relationship with the wave speed c .

- (a) Show that one can have traveling wave solutions to the heat equation $u_t = cu_{xx}$ by plugging in the traveling wave form and finding f . Note that σ can be arbitrary (in contrast to the wave equation)!
- (b) Looking at the general solution to the Cauchy problem for the heat equation (the convolution of the initial data with the heat kernel), describe the “domain of dependence” (in a sentence) for the solution u at a point $(\hat{x}, \hat{t})$ in spacetime. (At what “speed” does information seem to propagate via the heat equation?)

7. **2-D Hypothesizing.** Suppose that both $w(x, t)$ and $v(x, t)$ are two solutions to the 1-D heat equation $u_t = Du_{xx}$.

- (a) Show that the function $\pi(x, y, t) = w(x, t)v(y, t)$ is a solution to the 2-D heat equation $u_t = D\Delta u$ (recall that $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ where n is the dimension of the problem).
- (b) Venture a guess (using a double integral) at the solution of the Cauchy problem

$$u_t = D\Delta u, \quad u(x, y, 0) = \psi(x, y).$$

8. (Bell 9.1; Strauss 3.3) **Duhamel’s Principle, Again.** Review HW 4 on Duhamel’s Principle and Bell’s Section 9.1. Consider the inhomogeneous heat equation:

$$u_t - Du_{xx} = F(x, t), \quad u(x, 0) = 0.$$

- (a) Use the Duhamel procedure to obtain the formula $u(x, t) = \int_0^t \int_{-\infty}^{\infty} S(x - y, t - \tau) F(y, \tau) dy d\tau$ for the solution of the inhomogeneous equation above with zero initial data. (Here $S(x, t)$ is the standard heat kernel.)
- (b) Suppose that $u_1(x, t)$ solves $u_t - Du_{xx} = F(x, t)$, $u(x, 0) = 0$, and $u_2(x, t)$ solves $u_t - Du_{xx} = 0$, $u(x, 0) = \phi(x)$. What is the solution to the Cauchy problem $u_t - Du_{xx} = F(x, t)$, $u(x, 0) = \phi(x)$? Justify your answer.
- (c) Solve the following inhomogeneous problems:
 - i. $u_t - u_{xx} = xt$, $u(x, 0) = 0$.
 - ii. $u_t - u_{xx} = 2x \sin(t)$, $u(x, 0) = e^{-|x|}$.
 - iii. $u_t - u_{xx} = 3$, $u(x, 0) = H(x)$.

9. **Ill-Posed Problem.** Consider the backward heat equation $u_t + u_{xx} = 0$.

- Show that each member of the family $u_n(x, t) = 1 + \frac{1}{n}e^{n^2 t} \sin(nx)$ is a solution ($n = 1, 2, 3, \dots$).
- What is $\lim_{n \rightarrow \infty} u_n(x, 0)$? Show your work.
- Let $t = T$ be fixed. What happens to $u_n(x, T)$ as $n \rightarrow \infty$?
- Deduce the backward heat equation is not *well-posed* as a Cauchy problem and explain in a sentence or two.

10. (Bonus) **Boundedness and Asymptotics.**

- Suppose that the initial profile for the heat equation has the property that $|\phi(x)| \leq M$ for all x , where M is some positive constant. Show that the corresponding solution to the Cauchy problem satisfies $|u(x, t)| \leq M$ as well for all $x \in \mathbb{R}$ and all $t > 0$.
- Suppose that $\phi(x) = e^{-x}$ for all $x \in \mathbb{R}$. Is the solution bounded, as above? Does your answer bother you—why or why not? (See your answer from 5.(d).)
- (Strauss 2.4, #12) **Asymptotics.** Consider the Cauchy problem for the heat equation with the initial state $\phi(x) = H(x)$.

Let x_0 be fixed. Approximate the temperature $u(x_0, t)$ for large values of t using a power series expansion for e^{-z^2} and integrating term by term. (It may be helpful to remember that $\text{erf}(z)$ is a function you found a power series representation of in Calculus II.)

11. (Bonus) (Strauss 2.4, #14) Suppose now that the initial profile ϕ has the property that $|\phi(x)| \leq Ce^{ax^2}$. Show that there exists a time T such that the solution makes sense for $0 < t < T$, but not necessarily after T .

12. (Bonus) (Bell p. 1, #3; Strauss, p. 53 #9 and #10) **Polynomial Solutions.** There are polynomial solutions to the heat equation explored here, but note that these solutions are *not bounded at infinity*. **In this problem, assume that $D = 1$.**

- Show that $p(x, t) = x^4 + 12x^2t + 12t^2$ solve the heat equation with IC $\phi(x) = x^4$.
- Show that a function of the form $u(x, t) = A(t)x^2 + B(t)x + C(t)$ solves the heat equation with initial condition of $\phi(x) = x^2$. (By plugging in, solve for the time-dependent coefficients.)
- Solve the heat equation with the initial data $\phi(x) = x^2$, using the integral (convolution) form. Give your answer in integral form (do not try to integrate it in closed form), but make the substitution $p = (4Dt)^{-1/2}(x - y)$.
- Since the solution to the heat equation Cauchy problem *must be unique*, deduce the value of $\int_{-\infty}^{\infty} p^2 e^{-p^2} dp$.

13. (Bonus) **Transformations.** (Bell 8, #6) Show that Burgers equation $u_t = Du_{xx} - uu_x$ can be transformed into the heat equation $v_t = v_{xx}$ with the famous Cole-Hopf transformation:

$$\begin{cases} u = \psi_x \\ \psi = -2D \ln v. \end{cases}$$

(Note: This is substantial, since this transformation allows us to reduce a nonlinear evolution equation to a linear evolution that we can explicitly solve.)