September 22, 2025

Homework 4: Waves...Man

MATH 404, Fall 2024

Due: $\sim$ Monday, 10/6 in class.

This assignment is about the wave equation on $\mathbb{R}$ (and associated topics).

1. **Basic Wave Solution**. For each of the following IVPs, give the solution.
   (a) $u_{tt} - 9u_{xx} = 0, \quad u(x,0) = \sin(x), \quad u_t(x,0) = 0.$
   (b) $u_{tt} - 4u_{xx} = 0, \quad u(x,0) = 0, \quad u_t(x,0) = \frac{1}{1+(x/2)^2}.$
   (c) $u_{tt} - c^2u_{xx} = 0, \quad u(x,0) = \ln(1+x^2), \quad u_t(x,0) = 4+x.$

2. (Strauss, Section 2.1, #10) **Operator Factoring**.
   Solve the IVP below using *operator factoring* as we did for the wave equation:
$$u_{xx} + u_{xt} - 20u_{tt} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

3. **Domain of Dependence/Influence**.
   (a) (Strauss, Section 2.1, #3) The midpoint of a piano strong of constant tension T, density ρ, and (long) length l is hit (symmetrically) by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$.) How long does it take for the disturbance to reach the flea?
   (b) Consider the wave equation $u_{tt} - (1/4)u_{xx} = 0$ with initial velocity zero $u_t(x,0) \equiv 0$ and initial displacement $u(x,0) = \phi(x)$. Suppose that ϕ is *supported* in the set $[2,4] \cup [10,14]$. (Recall, the support of a function f is the *closure* of the set on which f does not vanish.)
      i. Give a rough sketch of the domain of influence for the initial condition ϕ.
      ii. Consider the graph of the domain of influence as a function of t (as t increases). At what value of t does the domain of influence become a connected set?
      iii. For each of the following points (x,t) in space-time, determine the intersection of the domain of dependence of that point and the support of the initial condition.
      A.) $(3,2)$ B.) $(6,1)$ C.) $(7,8)$ D.) $(0,26)$

4. (DuZ, pp.161–162) **Variable Changes**. The Telegraph Equation (see Bell Chapter 7), given by
$$u_{tt} - c^2u_{xx} + \delta u_t + m^2u = 0,$$
   is an example of the wave equation with all possible lower order terms. Consider
$$u_{tt} = a_1u_{xx} + a_2u_x + a_3u_t + a_4u. \tag{1}$$
   (a) Introduce the variable change $u(x,t) = \exp(\alpha x + \beta t)v(x,t)$ and plug this form into (1). Simplify and present the resulting PDE.
   (b) Since the coefficients α, β are arbitrary, show that you can choose them (based on a_i, $i = 1,2,3,4$) to eliminate the lower order terms involving u_t and u_x.
   (There is nothing special about which two lower order terms we eliminate. However, whatever lower order term is left, one then has to solve *that* equation....)

5. (Strauss, Section 2.2, #1 and #5; Bell 7, #2) **Energies**.
   (a) Consider the "plucked string" initial condition supported on the interval $[-a,a]$
$$\phi(x) = \begin{cases} b - \frac{b|x|}{a} & |x| < a \\ 0 & |x| > a \end{cases},$$
   for the standard wave equation on $\mathbb{R}$ with wave speed c. For some $T > 0$, compute $E(T)$.

(b) Consider the damped wave equation:

$$u_{tt} + \delta u_t - c^2 u_{xx} = 0.$$

Derive the corresponding *energy identity* here by multiplying by u_t and integrating in time and space (see class notes). You may assume at all times t, the solution is compactly supported. Then deduce that the total energy for the dynamics must decrease as t increases.

(c) Consider the Klein-Gordon equation:

$$u_{tt} - c^2 u_{xx} + m^2 u = 0.$$

Define a modified energy functional

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[c^2 (u_x(\xi,t))^2 + (u_t(\xi,t))^2 + m^2 (u(\xi,t))^2 \right] d\xi.$$

Show that the energy, in this sense, is conserved for any solution to the Klein-Gordon equation. (You may assume at all times t, the solution is compactly supported.)

6. (Bell, Chapter 5, #6) **Klein-Gordon**. Consider the (dispersive) *Klein-Gordon* equation:

$$u_{tt} - c^2 u_{xx} + m^2 u = 0.$$

(a) Consider a possible solution of the form $u(x,t) = F(x \pm ct)$, for F an arbitrary function. Show that such a solution cannot exist (unless $F \equiv 0$).

(b) Now try a modified solution: $u(x,t) = F(x + \sigma t)$, for $\sigma \in \mathbb{R}$, $\sigma \neq c$. Determine the form of solutions, not worrying about the initial conditions or any arbitrary constants that show up. (Be careful! You will have different cases, depending on the size of σ.)

(c) From the previous part, note that for certain values of σ we have oscillatory, bounded, wave solutions. Since σ is a wave speed, we note that there are waves (solutions) that propagate at speeds other than c (*faster than* c)—this is in contrast to the wave equation (see d'Alembert's formula).

To investigate this further, we return to our discussion of dispersion relationships. The dispersion relation for the Klein-Gordon equation here is $\omega = \pm\sqrt{c^2k^2 + m^2}$, where ω is the frequency of a given wave solution and k is the wave number.

i. Verify that $u(x,t) = A\cos(kx - \omega t)$ satisfies the Klein-Gordon equation above if k and ω satisfy the dispersion relationship.

ii. Noting that $kx - \omega t = k\left(x - \frac{\omega}{k}t\right)$, determine the wave speed $c(\omega_0)$ associated with a given frequency ω_0.

7. **Duhamel's Principle and Superposition**. In this question we consider the inhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \end{cases} \tag{2}$$

(a) Suppose u_1 satisfies (2) with $\phi = \psi \equiv 0$; suppose u_2 satisfies (2) with $F \equiv 0$ and $\phi \equiv 0$; suppose u_3 satisfies (2) with $F \equiv 0$ and $\psi \equiv 0$.

Explain, in detail, how to construct a "full" solution to the inhomogeneous IVP (2).

(b) We will recall Duhamel's principle for ODEs.

Consider $\tau \in \mathbb{R}$ to be a parameter here. Demonstrate that $y(t) = \displaystyle\int_0^t w(t-\tau,\tau)d\tau$ solves

$$y'(t) + ay = F(t),\ t > 0;\ y(0) = 0,$$

where the function $w(t,\tau)$ solves the (τ-parametrized) homogeneous problem with nonzero initial data:

$$w'(t;\tau) + aw(t;\tau) = 0,\ t > 0;\quad w(0;\tau) = F(\tau).$$

(c) Suppose that $w = w(x,t;\tau)$ is a solution to the τ-parametrized wave equation

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0;\tau) = 0, \quad w_t(x,0;\tau) = F(x,\tau) \end{cases}.$$

(Note, one can obtain a solution to this wave equation using d'Alembert's approach.)
Show that the function $u(x,t) = \int_0^t w(x, t-\tau; \tau)d\tau$ is a solution to

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad u(x,0) = u_t(x,0) = 0.$$

(d) Solve the following inhomogeneous Cauchy problems.

i. $u_{tt} - c^2 u_{xx} = xt; \quad u(x,0) = 0, \quad u_t(x,0) = 0$

ii. $u_{tt} - c^2 u_{xx} = \cos(x); \quad u(x,0) = 0, \quad u_t(x,0) = (1+x^2)^{-1}$

iii. $u_{tt} - c^2 u_{xx} = e^{2x}; \quad u(x,0) = \sin(x), \quad u_t(x,0) = 0$

8. **ODE Problems**.

(a) Use ODE methods to solve the following two problems in the solution variable $x(t)$:

i. $\begin{cases} \ddot{x} + x = \delta(t) \\ x(0) = 0, \quad \dot{x}(0) = 0 \end{cases}$

ii. $\begin{cases} \ddot{x} + x = 0 \\ x(0) = 0, \quad \dot{x}(0) = 1 \end{cases}$

where $\delta(t)$ is the "Delta function" (or unit impulse) centered at $t = 0$.

(b) Make an observation and try to interpret your conclusion from the physical meaning of these ODE systems (2 sentences).

(c) Try to relate this exercise with Problem 7bc. In particular, can you try to explain where Duhamel's principle comes from? What is the procedure "doing" to obtain the non-homogeneous solution from the homogeneous solution? Hint: `https://en.wikipedia.org/wiki/Duhamel%27s_principle`

BONUS

9. (Strauss, Section 2.1, #7) **Space Oddity**. Show that if both the initial disturbance $\phi(x)$ and the initial velocity profile $\psi(x)$ are odd functions, then a corresponding solution to the wave equation must also be an odd function (of x) for every t.

10. (Strauss, Section 2.1 and Section 2.1 #3) **Solving and Solutions**.

(a) Explain why smooth solutions to the transport IVP $u_t + cu_x = F(x,t), \quad u(x,0) = u_0(x)$ (for c a constant and u_0 a smooth function on $\mathbb{R}$) should be *unique*.

(b) Consider the system

$$\begin{cases} (\partial_t + c\partial_x)v = 0 \\ (\partial_t - c\partial_x)u = v \\ v(x,0) = \lambda(x) \\ u(x,0) = \phi(x) \end{cases} \tag{3}$$

Show that one has a solution to this system *if and only if* one can solve the wave equation with wave speed c, and initial disturbance $\phi(x)$ and initial velocity profile $\psi(x)$.

(c) Deduce (argue) that all solutions to the wave equation IVP on $\mathbb{R}$

$$\begin{cases} (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{4}$$

must take on the form of d'Alembert's solution.

11. (Strauss, Section 2.1 and 2.2) **Spherical Waves and Attenuation**. The three dimensional wave equation in solution variable $u(\vec{x}, t) = u(x, y, z, t)$ is given by

$$u_{tt} = c^2 \Delta u.$$

(The Laplacian is spatial, i.e., $\Delta = \sum_{i=1}^{3} \partial_{x_i}^2$.) If we assume a spherically symmetric solution $u(r, t)$, where $r > 0$ is the radial value, the equation we must then solve is

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right).$$

(a) Change variables using $v = ru$ to obtain a new equation for v.

(b) Solve for v, and transform back to a solution for u (thereby solving the spherical wave equation in u).

(c) What happens to the solutions as $r \to \infty$? Does this make sense from the point of view of what we know about waves and energies? Explain.

(d) In general, in n dimensions, a spherical wave satisfies the equation

$$u_{tt} = c^2 \left(u_{rr} + \frac{n-1}{r} u_r \right). \tag{5}$$

Consider a wave that has the form $u(r, t) = \alpha(r) f(t - \beta(r))$. (The function $\alpha(r)$ is called the attenuation, and $\beta(r)$ is called the delay.) The question is whether or not such solutions exist for *arbitrary functions* f. Existence of solutions means that distortionless spherical waves with attenuation are possible.

i. Plug in the special form of the solution into the spherical wave equation (5) to get an ODE for f.

ii. Set the coefficients of f'', f', and f equal to zero. Solve the resulting ODEs to see that (unless $u \equiv 0$) $n = 1$ and $n = 3$ are the only possibilities.

iii. Show that if $n = 1$, $\alpha(r)$ must be constant (no attenuation possible). (Thus $n = 3$ is the only dimension where one can have distortionless spherical wave propagation with attenuation.)