September 7, 2025

Homework 3: First Order

MATH 404, Fall 2025

Due: Monday, 9/22, in class (or by email to Madison by the end of the day on T, 9/23).

This assignment is about first order PDE.

1. (Bell, Section 3, p.8) **Modeling.**

   (a) In the 1-D derivation of the differential form of the conservation law (Section 3.1), what would the analogous result be if the cross-sectional area A were a smooth function of x, i.e., $A(x)$? Give your answer as a PDE.

   (b) In the pure advection case, $\rho_t + 5\rho_x = 0$, suppose the initial profile is $\rho_0(x) = e^{-x^2}$ (a Gaussian "bump"). Where (in space) is the global maximum of the solution $t = 10$? What is the global maximum? (Don't solve!)

2. **Advection with Decay**. Consider the equation $u_t + cu_x = -\lambda u$, $\lambda > 0$, with a given initial profile $u(x, 0) = u_0(x)$, where u_0 is a smooth, *bounded* function of x. Solve the problem using the method of characteristics, and explain in a sentence what happens as $t \to \infty$ and why.

3. **Basic Method of Characteristics.** Each of the problems below is given as an IVP with a profile in x, specified at t or $y = 0$. Solve using the method of characteristics, showing all details in each problem.

   (a) $u_t + 2u_x = 0, \quad u(x, 0) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$,

   (b) $\rho_t + (x\rho)_x = 0, \quad \rho(x, 0) = \sin(x)$

   (c) $u_t + (2xt)u_x = u, \quad u(x, 0) = x$

4. (DuZ 7.1, #2) **Influence of the Boundary I**. Consult DuZ pp.291–292 and solve the initial boundary value problems below completely.

   (a) $\begin{cases} u_x + u_t = 0, & x > 0, \quad t > 0 \\ u(x, 0) = e^{-x^2}, & x > 0 \\ u(0, t) = 1, & t > 0 \end{cases}$

   (b) $\begin{cases} u_x + u_t = 0, & x > 0, \quad t > 0 \\ u(x, 0) = 0, & x > 0 \\ u(0, t) = \sin(t), & t > 0 \end{cases}$

5. **Influence of the Boundary II**. Consider the initial boundary value problem

   $$\begin{cases} u_t + 2u_x = 0, & x > 0, \ t > 0 \\ u(x, 0) = e^{-x}, & x > 0 \\ u(0, t) = 1/(1 + t^2), & t > 0 \end{cases}$$

   The so called separating characteristic is $x = 2t$. For the region $x > 2t$, the solution comes from the initial condition; for the region $0 < x < 2t$ we apply the boundary condition.

   (a) Compute the solution in both of the regions described above and show that the parts agree along the lead characteristic $x = 2t$.

   (b) Compute the first partials in both regions, and again evaluate on the lead characteristic. Explain what is happening. (Note: "discontinuities are carried along characteristics".)

6. **Inviscid Burger**. Consider the quasilinear initial value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}.$$

(a) Describe the characteristics for the problem above, and group your answer in the x regions: $x < 0,\ 0 \leq x \leq 1,\ x > 1$.

(b) What happens at the point $(2, 1)$ in the x-t plane? Can a proper, smooth solution to the PDE exist for values of $t > 1$? Explain.

(c) Give the solution of the IVP for $t < 1$. (It will help to break the plane into three regions: (i) $x < 2t$, (ii) $x > t + 1$, (iii) $2t < x < t + 1$.)

7. **Implicit Solution**. Consider the quasilinear equation $u_t + c(u)u_x = 0$, where c is a smooth function. Show that the implicit expression

$$u = F(x - c(u)t), \quad F \text{ smooth}$$

defines a solution $u = u(x, t)$ (implicitly, of course) when it exists.

8. (DuZ Section 7.4) **Fans and Shocks**. Consider the IVP: $u_t + c(u)u_x = 0, \quad u(x, 0) = f(x)$

(a) Let $c(u) = u$ and show that the characteristics are straight lines.

(b) Let $c(u) = u$ and $f(x) = \begin{cases} x+1 & x < 0 \\ x+2 & x > 0 \end{cases}$. Recall the implicit solution $u = (x - ut) + 1$ in the region covered by characteristics emanating from the negative x-axis ($x < 0$), and a similar implicit description for characteristics starting from characteristics starting on the positive x axis ($x > 0$). Give the full *expansion fan* solution for this IVP.

(c) Let $c(u) = u$ and $f(x) = 2 - H(x)$, where H is the Heaviside function.

i. Describe the problematic region in the x-t plane.

ii. Give the *shock solution* there, or, at least describe what it is—see DuZ.

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9. (Bonus) **Initial Curve Γ.** For this problem, refer to the examples in Bell Section 4.3 and DuZ Section 7.2. Note that the initial curve must be parametrized, and your method needs to be more general than our approach in class. Make sure to state the region where the solution is valid.

(a) $(x + 2)u_x + 2yu_y = 2u, \quad u(-1, y) = y^{1/2}, \quad y > 0$

(b) $u_t + u_x = 1, \quad u(x, 2x) = f(x), \quad f \in C^1(\mathbb{R})$

(c) $t^2 u_t - x^2 u_x = 0, \quad u(1, t) = f(t), \quad f \in C^1(\mathbb{R})$

10. (Bonus) **Issues**. (Bell, Section 3, #8) Consider $yu_y + u_x = 0, \quad u(x, 0) = f(x)$.

(a) If $f(x) = x$, show no solution can exist.

(b) If $f(x) = 1$, show many solutions exist.

(c) Explain, as best you can, what is going wrong here.

(d) The initial curve above is $\{(x, y) = (s, 0) \ : \ s \in \mathbb{R}\}$. Can you modify the initial curve to guarantee that a solution exists?

11. (Bonus) Consider again the quasilinear IVP: $u_t + c(u)u_x = 0, \quad u(x, 0) = f(x)$

(a) Argue that if $c(u)$ and $f(x)$ are both nonincreasing or both nondecreasing then no shocks develop for $t \geq 0$.

(b) Let $c(u) = u$ and $f(x) = e^{-x^2}$. Find the point (x_b, t_b) in space-time where the wave *breaks*.