7 b.) Equation: $\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \begin{cases} x+1 & x<0 \\ x+2 & x>0 \end{cases} \end{cases}$

Implicit solution: $u = u_0(x-ut)$

$$\Rightarrow u = \begin{cases} x-ut+1 & x-ut<0 \quad \text{A} \\ x-ut+2 & x-ut>0 \quad \text{B} \end{cases}$$

$$\Rightarrow \quad u = x-ut+1 \qquad x-ut<0$$

$$\Rightarrow \quad (1+t)u = x+1 \Rightarrow \quad u(x,t) = \frac{x+1}{t+1}$$

A Region: $x-ut<0 \Rightarrow x-\left(\frac{x+1}{t+1}\right)t<0$

$$\Rightarrow (t+1)x-(x+1)t<0$$

$$\Rightarrow x-t<0 \Rightarrow x<t$$

$$u = x-ut+2 \Rightarrow u(x,t) = \frac{x+2}{t+1}$$

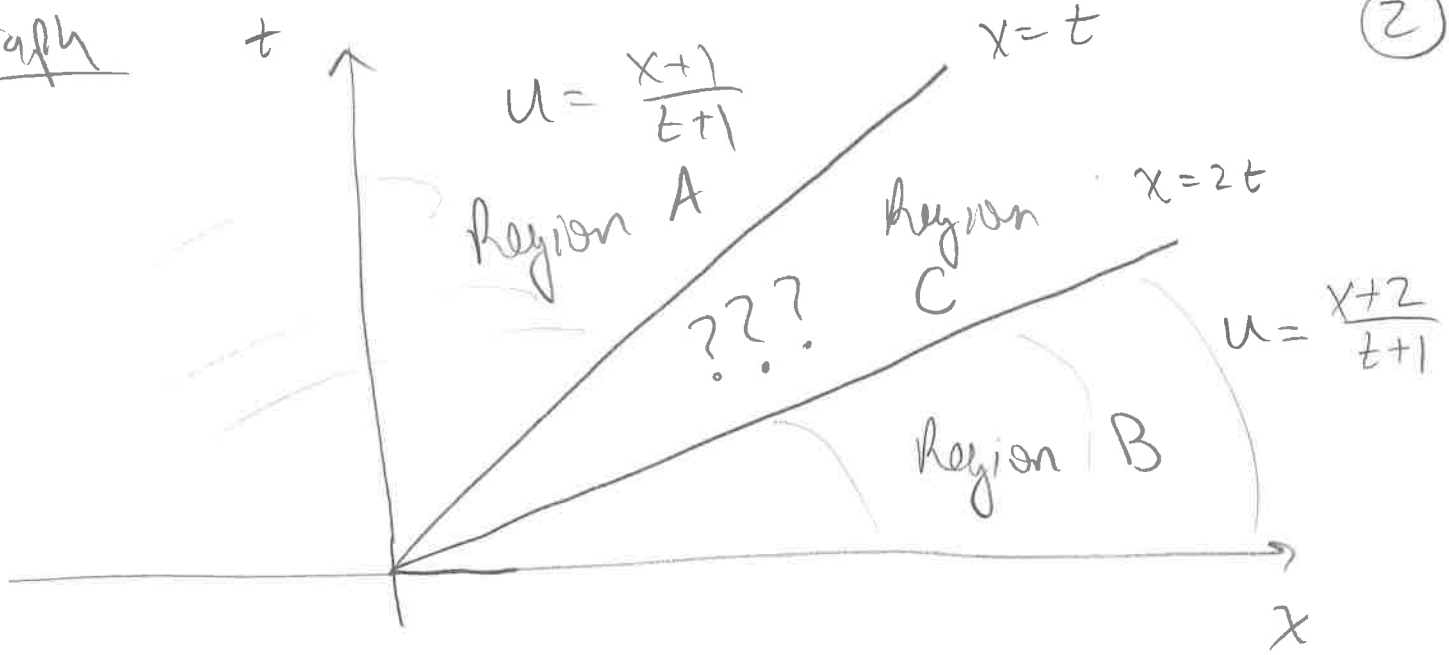
B Region: $x-ut>0$

$$\Rightarrow \quad x-\left(\frac{x+2}{t+1}\right)t>0$$

$$\Rightarrow (t+1)x-(x+2)t>0$$

$$x>2t$$

Graph



Need solution in Region C

Use DuZ p.309

sol^n in C is $g(x/t)$ where

$g = c^{-1}$ in $u_t + c(u)u_x = 0$

For $c(u) = u$, $c^{-1}(u) = u$, so

the sol^n is x/t. (Can check:

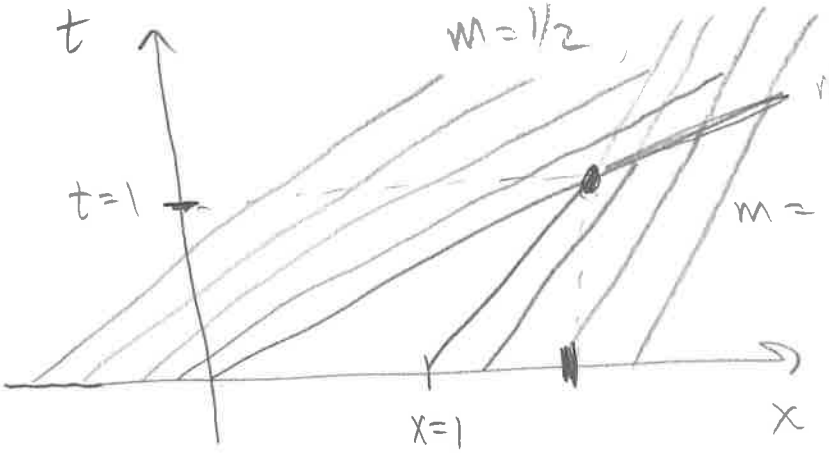
$\partial_x(x/t) = \frac{1}{t}$, $\partial_t(x/t) = -\frac{x}{t^2}$

so $\left[\frac{x}{t}\right]_t + \left[\frac{x}{t}\right]\cdot\left[\frac{x}{t}\right]_x = \frac{-x}{t^2} + \frac{x}{t^2} = 0$ ✓

So

$$u(x,t) = \begin{cases} \frac{x+1}{t+1} & x < t \\ x/t & t \le x \le 2t \\ \frac{x+2}{t+1} & x > 2t \end{cases}$$

c) For $u_0(x) = \begin{cases} 2 & x \le 0 \\ 1 & x > 0 \end{cases}$



$t = \frac{1}{2}x$

$t = x - 1$

The problematic region is the wedge $\{\frac{1}{2}x < t < x-1\}$. Characteristics double cover this region.