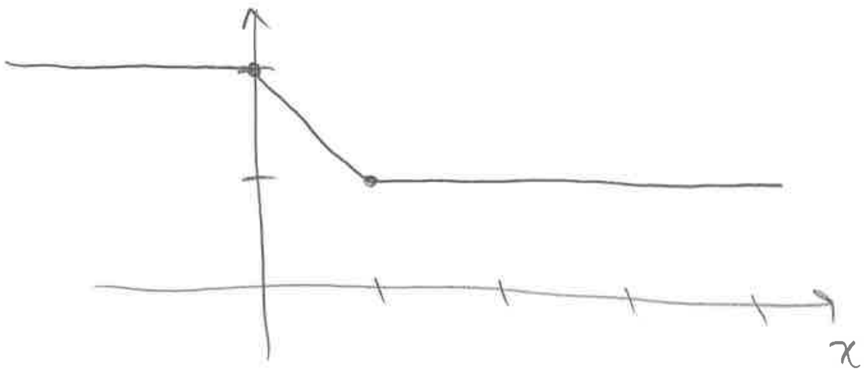
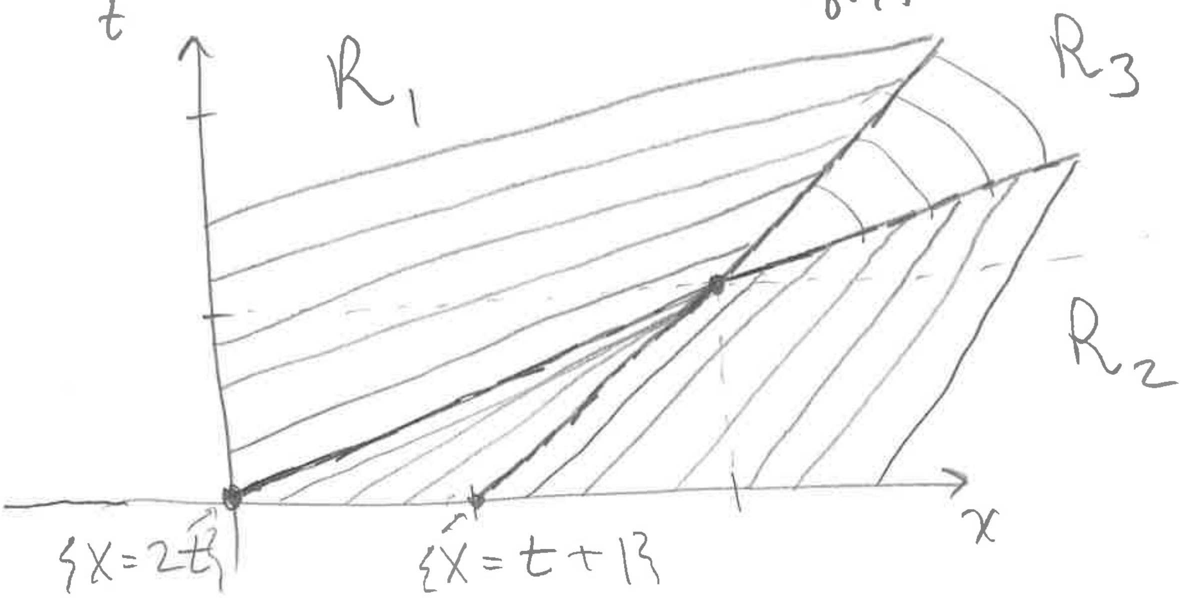
6. $$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 2 & x < 0 \\ 2 - x & x \in [0,1] \\ 1 & x > 1 \end{cases} \end{cases}$$

$\underline{u_0(x)}$



Characteristics

lines of slope $\frac{1}{u_0(\xi)}$



a) R_3 is the problematic region. This region is covered by multiple characteristics. Since RHS $= 0$, solution must be constant on characteristics, but if two characteristics intersect, they carry different values. Hence the solution, as a function, cannot exist there.

$$R_3 = \left\{ \frac{x}{2} < t < x-1,\ t > 1 \right\}$$

b) At $(2,1)$, many characteristics intersect. The solution cannot exist there (see above).

c) From class, implicit solution holds:

$u = u_0(x-ut)$ so

$$u = u_0(x-ut) = \begin{cases} 2, & x-ut < 0, \quad R_1 \\ 2-[x-ut], & 0 \leq x-ut \leq 1, \quad R_2 \\ 1, & x-ut > 1, \quad R_3 \end{cases}$$

Simplify for R_2:

$$u = 2 - [x - ut] \Rightarrow u = \frac{2-x}{1-t}$$

Region R_2:
$$x - ut = x - \left[\frac{2-x}{1-t}\right]t$$
$$= \frac{x - xt - 2t + xt}{1-t}$$
$$= \frac{x-2t}{1-t}$$

So $0 \le x - ut \le 1$

$$\Rightarrow 0 \le x - 2t \le 1 - t$$
$$\Rightarrow 2t \le x \le 1 + t$$

<u>So</u>

$$u(x;t) = \begin{cases} 2 & x < 2t \\ \dfrac{2-x}{1-t} & 2t \le x \le t+1 \\ 1 & x > t+1 \end{cases}$$