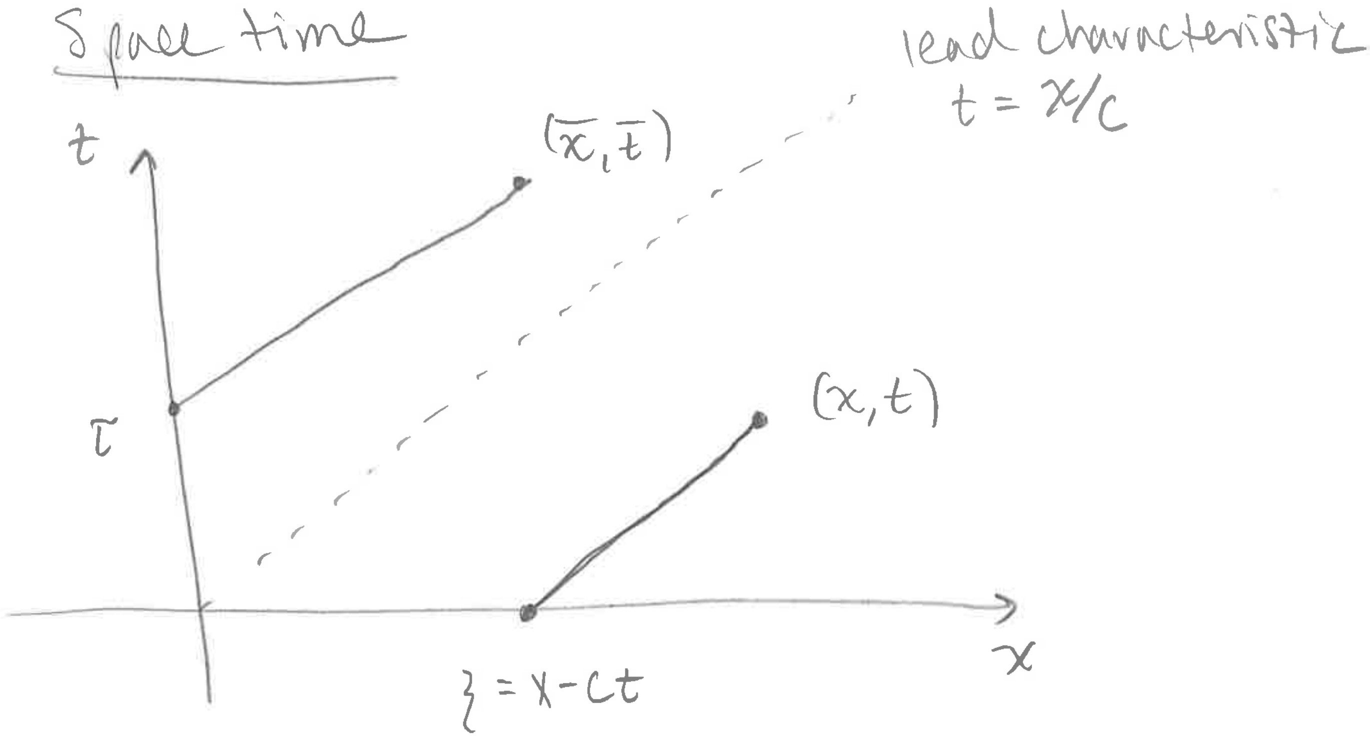
4) $\begin{cases} u_t + c u_x = 0 & c > 0 \\ u(x,0) = u_0(x) & x > 0 \\ u(0,t) = g(t) & t > 0 \end{cases}$

If $u_0(0) = g(0)$ we say the compatability condition holds.

Space time



Write τ in terms of $(\bar{x}, \bar{t})$:

$(0,\tau)$ + $(\bar{x}, \bar{t})$ connected by a line slope $\frac{1}{c}$

$\bar{t} - \tau = \frac{1}{c}\bar{x} \Rightarrow \tau = \bar{t} - \frac{\bar{x}}{c}$

Since RHS $=0$, $u = \text{const.}$ on characteristics

So

$$u(x,t) = \begin{cases} g\left(t - \frac{x}{c}\right) & t > x/c \quad \text{above lead} \\ u_0(x-ct) & t < x/c \quad \text{below lead} \end{cases}$$

on $x = ct$:

$$u(x,t) = \begin{cases} g(0) = u_0(0) & \text{when compatibility holds} \\ \text{DNE} & \text{when } g(0) \neq u_0(0) \end{cases}$$

<u>Derivatives</u>

$$\partial_t[u_0(x-ct)] = u_0'(x-ct)\cdot(-c)$$

$$\partial_x[u_0(x-ct)] = u_0'(x-ct)\cdot(1)$$

$$\partial_t[g(t-x/c)] = g'\left(t-\frac{x}{c}\right)$$

$$\partial_x\left[g\left(t-\frac{x}{c}\right)\right] = g'\left(t-\frac{x}{c}\right)\left(\frac{-1}{c}\right)$$

5b)

From a

$$u(x,t) = \begin{cases} \dfrac{1}{1+(t-\frac{x}{2})^2} & x<2t \\ e^{-(x-2t)} & x>2t \\ 1 & x=2t \end{cases}$$

$$\partial_t u(x,t) = \begin{cases} \dfrac{-2t+x}{\left(1+(t-\frac{x}{2})^2\right)^2} & x<2t \\ -e^{-(x-2t)}\cdot[-2] & x>2t \end{cases}$$

$$\partial_x u(x,t) = \begin{cases} \dfrac{t-\frac{x}{2}}{\left(1+(t-\frac{x}{2})^2\right)^2} & x<2t \\ -e^{-(x-2t)} & x>2t \end{cases}$$

So $$u_t = \begin{cases} 0 & x \nearrow 2t \\ 2 & x \searrow 2t \end{cases}$$

and $$u_x = \begin{cases} 0 & x \nearrow 2t \\ -1 & x \searrow 2t \end{cases}$$

$\Rightarrow$ jump discontinuities.