August 19, 2024

Homework 2: Mollifiers and Details

MATH 614, Fall 2024

Due: 9/18 in class

In the problems below, give a short writeup (length specified there) in LaTeX. .

1. **Mollifiers**. Read Appendix C.5 (including Theorem 7) in Evans. Read Kesavan 1.2 (up to and including Example 1.2.2—this is pp.4–5 in volume 3).

   Write up the definition and properties of *mollifiers*. You do not need to *prove* anything, but you should clearly explain what the main points are, and define any terms you use, e.g., what it means to *converge in a space* X.

   This should be about one page, LaTeXed.

2. **Details**. Write out each of the details from the proof(s) in Evans, including all justifications that Evans omits.

   (a) Suppose that $f \in C_0^2(\mathbb{R}^n)$. Prove—in detail—the differentiation formula

   $$u_{x_i} = \int_{\mathbb{R}^n} \Phi(\mathbf{y}) f_{x_i}(\mathbf{x} - \mathbf{y}) d\mathbf{y},$$

   for any $i = 1, 2, ..., n$.

   (b) Determine the integrability of the fundamental solution Φ (and its partial derivatives up to and including order two) *near the origin*. Note: your answer will depend on n.

   (c) Use polar coordinates (Evans p.715) to demonstrate

   $$\int_{\partial B_\epsilon(\mathbf{0})} |\Phi(\mathbf{y})| d\Gamma(\mathbf{y}) \leq C(\epsilon).$$

   In particular, find and prove the dependence on ϵ as in Evans' (14) on p.24. Again, your answer will depend on n.

3. **Regularity**. Write out Theorem 6 in Evans (p.28), and write up its (short) proof. Adding in some details that Evans omits. Try to justify each line in the calculation, as well as the conclusion: harmonic functions on an open set U have the property that they are $C^\infty(U)$.

   This should be about one page TeXed.

4. **Other Facts**. Write out Theorems 7–8 (inclusive) from Evans (pp.29–31) as well as their proofs. Including the details that Evans omits (justifications from line to line).