August 19, 2024

Homework 1: Classical PDEs

MATH 614, Fall 2024

Due: 9/9 in class

In the problems below, produce the classical solution (formula, procedure, etc.) to a given BVP/IVP/IBVP. Try to rigorously verify that it is satisfies the conditions of the problem (for the given general data). If you provide your derivation (and it is sound) that of course counts for a verification. Use your judgement about what needs to be justified and what can be stated, though you should try to justify convergence of series and integrals. You may of course ask me to clarify or for hints.

1. (Method of Characteristics) Consider the IVP on $\mathbb{R}$ in the solution variable $u(x,t)$ given by

$$\begin{cases} u_t + c(x,t)u_x = f(u,x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

   Assume that c and f are "nice" functions (i.e., you may assume anything reasonable you'd like, as long as you state what you assume).

   Describe how to obtain a formula/procedure for obtaining the solution.

   Hint: Assume that $x = x(t)$ and compute $\frac{d}{dt}[u(x(t),t)]$.

2. (Using the Laplace Transform) Solve the following IBVP problem for heat equation using the Laplace transform, showing all details:

$$\begin{cases} u_t - u_{xx} = 0 & \text{for } \ x > 0 \\ u(0,t) = g(t) & \text{for } \ t \geq 0 \\ u(x,0) = 0 & \text{for } \ x > 0. \end{cases}$$

3. (Heat Kernel and Reflection) Consider the half-space $\mathbb{R}^2_+ := \{\mathbf{x} \in \mathbb{R}^2 \ : \ x_2 > 0\}$. Let $D > 0$ and solve the homogeneous Neumann problem on the half space in $u(\mathbf{x};t)$:

$$\begin{cases} u_t - D\Delta u = 0, & \text{in } \mathbb{R}^3_+ \times (0,T) \\ u(\mathbf{x},0) = \phi(\mathbf{x}) & \text{in } \mathbb{R}^3_+ \\ u_{x_3}(x_1,x_2,0;t) = 0 & \text{on } \{x_3 = 0\} \times [0,T). \end{cases} \tag{2}$$

   Hint: You will need to assume some regularity for the initial data and a compatibility condition between the boundary and initial data. Begin by writing down the correct solution in the entire space $\mathbb{R}^2$, which you may cite in your solution construction. Then use the appropriate method of reflection.

4. (Laplace Equation)

   (a) Let Ω be a 2D bounded region bounded by a curve Γ, where Γ is a positively oriented, p.w. smooth, simple, closed curve in $\mathbb{R}^2$. (Note these are the hypotheses for the classical *Green's Theorem*.) Recall that function u is called *harmonic* if $\Delta u = 0$ (in any dimension).

      i. Argue that, if u is harmonic, then

$$\oint_\Gamma \nabla u \cdot \mathbf{n}\ ds = 0.$$

      ii. Argue that, if u is harmonic and $u(x,y) = 0$ on Γ, then

$$\iint_\Omega |\nabla u|^2\ dA = 0.$$

      What can you infer about u in this case?

(b) Consider Laplace's equation on the half space $\mathbb{H} \equiv \{(x,y) \ : \ y > 0\}$ for the solution variable $u(x,y)$ with Dirichlet boundary data:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 & \text{for} \quad (x,y) \in \mathbb{H} \\ u(x,0) = f(x). \end{cases}$$

Solve the BVP above by taking the Fourier transform in x and enforcing the extra conditions that $u(x,y)$ and $\hat{u}(\xi, y)$ stay bounded for all of their arguments.

(c) Consider the Dirichlet boundary data replaced by Neumann data:

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 & \text{for} \quad (x,y) \in \mathbb{H} \\ u_y(x,0) = g(x). \end{cases}$$

Solve the BVP by differentiating the PDE in y (but keeping the BC as is) and appealing to your solution from the previous part.

5. (Green's Function for Dirichlet Poisson) Consider $\Omega \subseteq \mathbb{R}^n$ to be a smooth domain with boundary Γ. Assume the data in this problem are smooth $g \in C^\infty(\Gamma)$ and $F \in C^\infty(\overline{\Omega})$. Use the fundamental solution for Laplace and the Green's function construction to give the solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for Poisson's equation with given Dirichlet data:

$$\begin{cases} -\Delta u = F, & \mathbf{x} \in \Omega \\ u\big|_\Gamma = g, & \mathbf{x} \in \Gamma. \end{cases} \tag{3}$$

Hint: Use Evans. Do not show all details leading up to the use of the fundamental solution. Use the principle of superposition, and be careful in defining the so called corrector function—you may assert that it exists (we will show that in this class), but in general it is very difficult to find explicitly.

6. (Wave Equation) In this problem you will explore the wave equation in 1,2, and 3 dimensions, under slightly different circumstances for each.

(a) Solve the 1-D inhomogeneous wave equation in $u(x,t)$ using the principle of superposition and variation of parameters (Duhamel's principle). You may assume whatever you need from the data ϕ, ψ and $F(x,t)$, but be clear in what you assume.

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & \text{in } \mathbb{R} \times (0,T) \\ u(x,0) = \phi(x); \ u_t(x,0) = \psi(x), & \text{in } \mathbb{R}. \end{cases} \tag{4}$$

Hint: Use d'Alembert on the appropriate subproblem after superposing. You will use d'Alembert again in solving the auxiliary problem associated to the Duhamel formula.

(b) Use the method of separation of variables to solve the homogeneous 2-D wave equation on the unit square $\mathbb{S} = (0,1)^2$ in $u(x_1, x_2; t)$, assuming $\phi \in C^2(\overline{\mathbb{S}})$:

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \text{in } \mathbb{S} \times (0,T) \\ u(\mathbf{x},0) = \phi(\mathbf{x}); \ u_t(\mathbf{x},0) = 0, & \text{in } \overline{\mathbb{S}} \\ u(\mathbf{x},t)\big|_\Gamma = 0 & \text{on } \Gamma = \partial\mathbb{S} \times [0,T). \end{cases} \tag{5}$$

Hint: Do the separation between t, x_1, and x_2, solve the relevant eigenvalue problems, and use Fourier theory. Note that the assumption on ϕ gives that ϕ and its partials up to order two are all in $L^2(\mathbb{S})$.

(c) Note the solution formula (Kirchhoff's) for the wave equation on $\mathbb{R}^3$ from Evans:

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \text{in } \mathbb{R}^3 \times (0,T) \\ u(\mathbf{x},0) = \phi(\mathbf{x}); \ u_t(\mathbf{x},0) = \psi(\mathbf{x}), & \text{in } \mathbb{R}^3. \end{cases} \tag{6}$$

For this one, please just state the formula then justify differentiation through integration to show that the solution solves the problem. This will require stating clear assumptions (as in Evans) for the data functions ϕ, ψ.