September 2, 2025

Homework 1: Review Material—Solutions

MATH 404, Fall 2024

1. Integration.

   (a) Compute, showing all work (not making use of a table of integrals)
   $\int_0^\infty (t-3)e^{-tx}dt$
   First consider the integral
   $$\int_0^M (t-3)e^{-tx}dt.$$
   Via integration by parts $u = (t-3), \quad dv = e^{-tx}$, we have
   $$\int_0^M (t-3)e^{-tx}dt = \frac{(t-3)}{x}e^{-tx}\Big|_{t=0}^{t=M} - \int_0^M \frac{-1}{x}e^{-tx}dt = \frac{(t-3)}{x}e^{-tx}\Big|_{t=0}^{t=M} - \frac{1}{x^2}e^{-tx}\Big|_{t=0}^{M}.$$
   Thus
   $$\int_0^M (t-3)e^{-tx}dt = \frac{(M-3)}{x}e^{-Mx} + \frac{3}{x} - \frac{1}{x^2}e^{-Mx} + \frac{1}{x^2}$$
   Taking the limit as $M \to \infty$ and using *l'Hospital* we arrive at the improper Riemann integral, and have
   $$\int_0^\infty (t-3)e^{-tx}dt = \frac{3}{+}\frac{1}{x^2},$$
   requiring that $x > 0$ (otherwise the integral doesn't converge).

   (b) Suppose that f, g are smooth functions that have the property that $f(0) = f'(0) = g(0) = g'(0) = 0$. Show that
   $$\int_0^L f^{(4)}(x)g(x)dx = \int_0^L f''(x)g''(x)dx + f'''(L)g(L) - f''(L)g'(L).$$
   We will use the version of integration by parts that reads as:
   $$\int_a^b f(x)g'(x)dx = f(x)g(x)\big|_{x=a}^{x=b} - \int_a^b f'(x)g(x)dx.$$
   We have
   $$\begin{aligned}\int_0^L f^{(4)}(x)g(x)dx &= f'''(x)g(x)\big|_{x=0}^{x=L} - \int_0^L f'''(x)g'(x)dx \\ &= f'''(L)g(L) - \int_0^L f'''(x)g'(x)dx \\ &\quad \text{invoking the BC } g(0) = 0 \\ &= f'''(L)g(L) - \left[f''(x)g'(x)\big|_{x=0}^{x=L} - \int_0^L f''(x)g''(x)dx\right] \\ &\quad \text{integrating by parts again and invoking the BC } g'(0) = 0 \\ &= f'''(L)g(L) - f''(L)g'(L) + \int_0^L f''(x)g''(x)dx]\end{aligned}$$

(c) Showing your work, compute each of the following integrals for all integers $1 \leq m, n \leq 3$:

i. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx$

ii. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx$

Let's adopt a convenient notation:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx \equiv (f, g).$$

For $(\sin(mx), \cos(nx))$ the associated 9 integrals can be computed using basic trig integrations (with trig identities like double and half angle formulas). Instead, let us note the product to sum formulae:

$$\begin{aligned}\sin(mx)\cos(nx) &= \frac{1}{2}\left[\sin((m+n)x) - \sin((m-n)x)\right] \\ \sin(mx)\sin(nx) &= \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right]\end{aligned}$$

When $m = n$, these collapse into the more familiar

$$\begin{aligned}\sin(mx)\cos(mx) &= \frac{1}{2}\sin(2mx) \\ \sin^2(mx) &= \frac{1}{2}\left[1 - \cos(2mx)\right]\end{aligned}$$

Each of the above can be integrated using elementary techniques (the trig terms integrate away on $[-\pi, \pi]$). Then, using our notation, we have (letting m dictate the row and n the column)

$$\left[(\sin(mx), \cos(nx))_{m,n=1,2,3}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[(\sin(mx), \cos(nx))_{m,n=1,2,3}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) The families

$$\{\sin(nx)\}_{n=1}^{\infty} \quad \text{and} \quad \{\cos(nx)\}_{n=1}^{\infty}$$

are orthogonal to one another. If we choose any function from either family and integrate it against any other in terms of (f, g) we get zero. If we integrate two distinct functions from the same family, those are also zero. The only time we get something other than zero (1, in fact) is when we form $(\sin(mx), \sin(mx))$ or $(\cos(nx), \cos(nx))$.

2. Series.

   (a) For each of the following series, determine: if it absolutely converges **AC**, if it converges but not absolutely **C** , or if it diverges **D**. Then state in a sentence or two how you arrived at your conclusion.

      i. $\displaystyle\sum_{n=1}^{\infty} \pi^{-n}$

      AC. This is a geometric series that converges since $\frac{1}{\pi} < 1$. In particular, it converges to $\dfrac{1}{\pi - 1}$. Since it is a positive series that converges, it converges absolutely.

      ii. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2 - 1}$

      D. This series diverges, for instance, using the limit comparison test or the direct comparison test with the harmonic series.

      iii. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

      AC. Note that $\left|\dfrac{\cos(n)}{n^2}\right| \leq \dfrac{1}{n^2}$. We have that $\sum \frac{1}{n^2}$ converges by the p-test (or the integral test), and thus $\sum \left|\frac{\cos(n)}{n^2}\right|$ converges by comparison.

3. Vector Operations.

   (a) The divergence of a vector field **F** at a point **x** measures the extent to which the vector field points inward (negative) or outward (positive), in aggregate, at that point. It is closely related to the *flux* of that vector field in any small region containing the point **x**.

   (b) Write out the product rule for the divergence:

$$\text{div } (u\vec{F}) = u(\text{div } \vec{F}) + \nabla u \cdot \vec{F},$$

   where u is a scalar function. (Checking it in 2-D is alright.)

   Let's use the convention that the vector $\vec{v}$ has components v_i. Note, then that $u\vec{F}$ is a vector with components $(u\vec{F})_i = uf_i$. Thus, when computing a partial we will be looking at $\partial_j(uf_i)$. Noting that both of these quantities are scalars, we invoke the scalar product rule (from Calc. 1).

$$\partial_j(uf_i) = (\partial_j u)f_i + u(\partial_j f_i).$$

   Thus

$$\text{div } (u\vec{F}) = \sum_j \partial_j(uf_j) = \sum_j \partial_j u f_j + \sum_j u\partial_j f_j = \nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F}),$$

   by the definition of the divergence.

4. Kernels of some operations.

   (a) Let $\vec{F}$ be a (smooth) conservative vector field with potential function ϕ. What must be true of curl $\vec{F} = \nabla \times \vec{F} = \nabla \times \nabla\phi$?

   In other words, the curl of a conservative vector field is...(finish this sentence and justify your claim.)

   The curl of a conservative vector field is the zero vector! We can check this by direct computation, using the equality of mixed partials. Indeed, since $\vec{F}$ is conservative, $\vec{F} = \nabla\phi$ for some scalar function ϕ. So $\vec{F} = \langle \phi_x, \phi_y, \phi_z \rangle$. Recalling that the curl of a 3-D vector field is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = (\partial_y f_3 - \partial_z f_2)\vec{i} - (\partial_x f_3 - \partial_z f_1)\vec{j} + (\partial_x f_2 - \partial_y f_1)\vec{k}.$$

   Now, in this case, this reduces to

$$\text{curl } \vec{F} = (\partial_y \phi_z - \partial_z \phi_y)\vec{i} - (\partial_x \phi_z - \partial_z \phi_x)\vec{j} + (\partial_x \phi_y - \partial_y \phi_x)\vec{k} = \vec{0}.$$

   (b) Let $\vec{C} = \nabla \times \vec{F}$. What must be true of div $\vec{C} = \nabla \cdot \vec{C} = \nabla \cdot (\nabla \times \vec{F})$?

   In other words, the divergence of a curl field must be...(finish this sentence and justify your claim.)

   The divergence of a curl field must be zero! We again check by direct computation using the above description of the curl and, again, the equality of mixed partials.

$$\nabla \cdot \text{curl } \vec{F} = \partial_x(\partial_y f_3 - \partial_z f_2) - \partial_y(\partial_x f_3 - \partial_z f_1) + \partial_z(\partial_x f_2 - \partial_y f_1) = 0$$

   after inspection.

5. Integration, II.

   (a) Such line integrals represent the area under a two dimensional surface when restricted to the path Γ. They are computed via a parametrization of the curve Γ and the calculation of the line element ds.

   (b) Compute the surface area (showing all work) of a spherical cap with height h cut from a sphere with radius ρ.

   In general, we are interested in the surface integral:

$$\iint_S (1)dS = \iint_T 1||\vec{r}_u \times \vec{r}_v||dudv,$$

   where S is the surface in question, and $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ is a parametrization of S over the region $(u, v) \in T$. (The symbol $\times$ is the cross product; $||\vec{v}||$ refers to the norm of the vector $\vec{v}$ in $\mathbb{R}^3$.)

   To compute a surface integral, we need to parameterize $\vec{r} = \langle x, y, z \rangle$ the upper hemisphere. This is a standard parametrization (coming from the Spherical to Cartesian change of variables—u is traditional angular variable in the x-y plane, and v is the polar angle, emanating from $\vec{k} \to -\vec{k}$):

$$\begin{cases} x(u, v) = & \rho \sin(v) \cos(u) \\ y(u, v) = & \rho \sin(v) \sin(u) \\ z(u, v) = & \rho \cos(v) \end{cases}$$

Thus (after some extensive calculation with calculus and trig identities):

$$\begin{aligned}
\vec{r}_u =& \langle -\rho\sin(v)\sin(u), \rho\sin(v)\cos(u), 0\rangle \\
\vec{r}_v =& \langle \rho\cos(v)\cos(u), \rho\cos(v)\sin(u), -\rho\sin(v)\rangle \\
\vec{r}_u \times \vec{r}_v =& \langle -\rho^2\sin(v)\cos(u), -\rho^2\sin^2(v)\sin(u) - \rho^2\sin(v)\cos(v) \\
||\vec{r}_u \times \vec{r}_v|| =& \rho^2\sin(v).
\end{aligned}$$

Now we need to determine what values of u, v between 0 and π parameterize the cap. If we allow all values of $0 \leq v \leq \pi$ we will get the hemisphere. Doing some trig reveals that for radius ρ and height h, the parameter range we are interested in is $0 \leq v \leq \arccos\left(1 - \frac{h}{\rho}\right)$. Thus the double integral we want is:

$$\int_0^{2\pi}\int_0^{\arccos(1-h/\rho)} \rho^2\sin(v)dvdu = -2\pi\rho^2[\cos(v)]_0^{\arccos(1-h/\rho)} = 2\pi\rho h.$$

6. Integration, III.

Recall that for any smooth, oriented curve $\Gamma \subset \mathbb{R}^2$ we can define the unit radial (with positive orientation) vector $\vec{r}(x,y)$ and the unit outward normal vector $\vec{n}(x,y)$. For all values along Γ, we have $d\vec{r}\cdot d\vec{n} = 0$, where $d\vec{r} = \langle dx, dy\rangle$ and $d\vec{n} = \langle dy, -dx\rangle$.

Consider C_1 to be the semicircle of radius two in the upper half plane with standard orientation (counter clockwise). Compute the following in any way you'd like.

(a) The average value of the function $f(x,y) = x + y + 2$ on C_1.
We define the average value to be

$$\frac{\overline{f} = \int_{C_1} f(x,y)ds}{|C_1|}.$$

Since C_1 is the semicircle, we know the length of $C_1 = 2\pi$. To compute the line integral, we utilize a standard parameterization with $x(t) = 2\cos(t),\ y(t) = 2\sin(t),\ \ t \in [0,\pi]$. Recalling the arc length element $ds = (dx^2 + dy^2)^{1/2}$, we can invoke the parametrization, as well as $x'(t) = -2\sin(t),\ \ y'(t) = 2\cos(t)$.

$$\int_{C_1}(x+y+2)ds = \int_0^{\pi}[2\cos(t) + 2\sin(t) + 2]\sqrt{4\cos^2(t) + 4\sin^2(t)}dt.$$

Invoking the pythagorean trig identity and integrating, we arrive at $8 + 4\pi$. Thus $\overline{f} = \frac{4+2\pi}{\pi}$.

(b) The flux through C_1 of the vector field $\vec{F}(x,y) = \langle x^2y\ ,\ (1/3)x^3 + y\rangle$

$$\int_{C_1}\vec{F}\cdot d\vec{n} = \int_{C_1}\langle x^2y, (1/3)x^3 + y\rangle\cdot\langle dy, -dx\rangle = \int_{C_1} x^2ydy - \left(\frac{1}{3}x^3 + y\right)dx$$

We now invoke the parametrization, as above: Using elementary integration, we have that the integral equals 2π.

$$\int_{C_1} x^2ydy - \left(\frac{1}{3}x^2 + y\right)dx = \int_0^{\pi}[(64/3)\cos^3(t)\sin(t) + 4\sin^2(t)]dt$$

(c) Now consider C_2 to be the line segment going from $(-2, 0)$ to $(2, 0)$. Compute the circulation of $\vec{F}$ (as given above) over the closed curve $C = C_1 \cup C_2$.

This is a closed curve, and one can quickly check that $\vec{F}$ is a conservative vector field. The circulation of a conservative vector field is always zero, by the fundamental theorem of line integrals.

7. The Laplacian.

(a) Recall that for $f(x, y)$ we have $\Delta f = \nabla \cdot \nabla f = f_{xx} + f_{yy}$. Consider polar coordinates with the change of variable mapping:

$$\begin{aligned} \theta(x, y) &= \arctan(y/x) \\ r(x, y) &= \left[x^2 + y^2\right]^{1/2}. \end{aligned}$$

Thinking of a f as $f(r, \theta)$, and using the *chain rule*, compute the expression for Δf in terms of r and θ.

Here we will also need to note the inverse relations:

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

We first note that by the chain rule

$$\partial_x f = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x},$$

which can be simplified by computing the partials of r, θ. After some work, we see that

$$r_x = \frac{x}{(x^2 + y^2)^{1/2}}, \qquad \theta_x = \frac{-y}{x^2 + y^2}.$$

Invoking the inverse relations above, we have

$$r_x = \cos(\theta), \quad \theta_x = \frac{-\sin(\theta)}{r}.$$

Thus the differentiation in x goes as

$$\frac{\partial f}{\partial x} = f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r}.$$

A similar calculation shows that

$$\frac{\partial f}{\partial y} = f_r \sin(\theta) + f_\theta \frac{\cos(\theta)}{r}.$$

We can then iterate to find the necessary second partials:

$$\partial_x^2 f = \partial_x(f_x) = \partial_x\left[f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r}\right].$$

$$\begin{aligned}
\partial_x^2 f &= \partial_x\left[f_r\cos(\theta) - f_\theta\frac{\sin(\theta)}{r}\right] \\
&= \partial_r\left[f_r\cos(\theta) - f_\theta\frac{\sin(\theta)}{r}\right]\cos(\theta) - \partial_\theta\left[f_r\cos(\theta) - f_\theta\frac{\sin(\theta)}{r}\right]\frac{\sin(\theta)}{r} \\
&= f_{rr}\cos^2(\theta) - f_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r} + f_\theta\frac{\sin(\theta)\cos(\theta)}{r^2} \\
&\quad + f_r\frac{\sin^2(\theta)}{r} - f_{r\theta}\frac{\cos(\theta)\sin(\theta)}{r} + f_{\theta\theta}\frac{\sin^2(\theta)}{r^2} + f_\theta\frac{\cos(\theta)\sin(\theta)}{r^2}
\end{aligned}$$

Similarly, in y, we have:

$$\begin{aligned}
\partial_y^2 f &= \partial_y\left[f_r\sin(\theta) + f_\theta\frac{\cos(\theta)}{r}\right] \\
&= \partial_r\left[f_r\sin(\theta) + f_\theta\frac{\cos(\theta)}{r}\right]\sin(\theta) + \partial_\theta\left[f_r\sin(\theta) + f_\theta\frac{\cos(\theta)}{r}\right]\frac{\cos(\theta)}{r} \\
&= f_{rr}\sin^2(\theta) + f_{r\theta}\frac{\sin(\theta)\cos(\theta)}{r} - f_\theta\frac{\sin(\theta)\cos(\theta)}{r^2} \\
&\quad + f_r\frac{\cos^2(\theta)}{r} + f_{r\theta}\frac{\cos(\theta)\sin(\theta)}{r} + f_{\theta\theta}\frac{\cos^2(\theta)}{r^2} - f_\theta\frac{\cos(\theta)\sin(\theta)}{r^2}
\end{aligned}$$

Putting it all together, and using the Pythagorean identity:

$$\Delta f = \partial_r^2 f + \frac{\partial_r}{r}f + \frac{\partial_\theta^2}{r^2}f = \frac{1}{r}\partial_r\left[r\partial_r f\right] + \frac{1}{r^2}\partial_\theta^2 f.$$

(b) Let Ω be a region bounded by a curve Γ, where Γ is a positively oriented, p.w. smooth, simple, closed curve in $\mathbb{R}^2$. (Note these are the hypotheses for *Green's Theorem.*) Recall that function u is called *harmonic* if $\Delta u = 0$ (in any dimension).

i. Argue that, if u is harmonic, then

$$\oint_\Gamma \nabla u \cdot \vec{n}\ ds = 0.$$

This is a direct application of the divergence theorem, using the description of the Laplacian as $\Delta = \text{div}\ \nabla$. Since we have assumed that $\Delta u \equiv 0$ on the region Ω, we can integrate this scalar function to arrive at

$$\iint_\Omega (\Delta u) dA = 0.$$

On the other hand, invoking the divergence theorem, we have:

$$\iint_\Omega (\Delta u) dA = \iint_\Omega \text{div}\ \nabla u\ dA = \oint_\Gamma \nabla u \cdot \vec{n} ds.$$

Putting the two identities together, we have the desired result.

ii. Argue that, if u is harmonic and $u(x, y) = 0$ on Γ, then

$$\iint_\Omega \nabla u \cdot \nabla u\ dA = 0.$$

What can you infer about u in this case?
Let's begin by recalling the relation (using the product rule from an earlier problem):

$$\text{div } (u\nabla u) = \nabla \cdot (u\nabla u) = \nabla u \cdot \nabla u + u(\nabla \cdot \nabla u) = |\nabla u|^2 + u(\Delta u).$$

Thus, integrating we have:

$$\iint_\Omega \text{div } (u\nabla u) dA = \iint_\Omega \nabla u \cdot \nabla u \; dA + \iint_\Omega u(\Delta u) \; dA.$$

We invoke the divergence theorem on the first term, as well as the hypothesis that $u \equiv 0$ on Γ:

$$\iint_\Omega \text{div } (u\nabla u) dA = \oint_\Gamma u(\nabla u \cdot \vec{n}) ds = \oint_\Gamma (0)(\nabla u \cdot \vec{n}) ds = 0.$$

On the other hand, since we assumed that u was harmonic, we know that $\Delta u \equiv 0$ in Ω, and thus

$$\iint_\Omega u(\Delta u) dA = \iint_\Omega u(0) dA = 0.$$

Putting these together, we have that

$$\iint_\Omega \nabla u \cdot \nabla u \; dA = 0.$$

(Since $\nabla u \cdot \nabla u = |\nabla u|^2 \geq 0$, we know that $|\nabla u| = 0$, and this can only happen if $u =$ constant. But since we know that $u = 0$ on Γ, we can conclude that $u \equiv 0$ on the entirety of $\overline{\Omega}$.)

8. First order ODE.

   (a) Verify that $\quad y(x) = x[1 + \cos(x)] \quad$ solves the IVP:

   $$\frac{dy}{dx} = \frac{y}{x} - x\sin(x), \quad y(\pi) = 0.$$

   For our purported solution, we have $\dfrac{dy}{dx} = [1+\cos(x)] - x\sin(x)$. In addition, we have that $\dfrac{y}{x} = [1+\cos(x)]$. So $\dfrac{dy}{dx} - \dfrac{y}{x} = -x\sin(x)$, which confirms that this y satisfies the ODE. We must also check the initial condition: $y(\pi) = \pi[1 + \cos(\pi)] = \pi[1 - 1] = 0$. So the initial condition is satisfied as well.

   (b) Solve the ODE

   $$t^3 y' + 4t^2 y = e^{-t}, \quad t > 0.$$

   Since $t > 0$, divide through by t^3 to obtain

   $$y' + \frac{4}{t} y = t^{-3} e^{-t}.$$

In this case (first order, linear ODE) we can invoke the integrating factor. Here, the integrating factor is $\mu = exp\left(\int 4t^{-1}dt\right) = e^{4\ln|t|} = t^4$. The ODE can then be written as: $(t^4y)' = te^{-t}$. We can then integrate (by parts) to arrive at

$$t^4y = -te^{-t} + e^{-t} + C,$$

which can be rewritten as

$$y(t) = t^{-4}e^{-t} - t^{-3}e^{-t} + C.$$

9. Modeling.

   Let $P(t) \geq 0$ be some population at time $t \geq 0$. Let $P(0) = P_0 \geq 0$. Consider the population model
$$\frac{dP}{dt} = 2P(100 - P)(P - 20), \quad P(0) = P_0.$$
   In this model $K = 100$ is the *carrying capacity* of the population and $M = 20$ is the *healthy population minimum.*

   Without solving or drawing a slope field (both are unpleasant), describe the properties of the solution in about 3 sentences. (What are the equilibrium solutions? What are the ultimate possible outcomes for the population (as $t \to \infty$)? How does the "shape" of a solution and its ultimate outcome depend on the initial data P_0?) Use your intuition, and see what the DE tells you directly. Think about what the IVP is *trying* to model.

   The equilibrium solutions are the values of P for which the function $P(t)$ is *stationary*, i.e., $P' = 0$. This occurs for $P = 0$ (no reproduction), $P = 100$ (at carrying capacity), and $P = 20$ (the healthy population minimum). If the initial population $P_0 \in (0, 20)$, there are not enough members of the population to successfully grow, and thus the population $P(t)$ tends toward zero (decreases). This can be seen since for such values of P, we observe that $\frac{dP}{dt} < 0$. When the initial population $P_0 \in (20, 100)$, the sign of $\frac{dP}{dt} > 0$, and thus the population $P(t)$ grows toward the carrying capacity as $t \to \infty$. If the population starts above the carrying capacity, $P_0 > 100$, then the sign of $\frac{dP}{dt} < 0$ again, and the population will decrease toward that equilibrium solution $P = 100$.

10. Second order constant coefficient ODE.

    (a) Consider the second order, constant coefficient differential equation in $x(t)$:
$$x'' + bx' + 2x = 0. \tag{1}$$
    Find a value for b so that solutions to this ODE are damped, but not overdamped—i.e., such that solutions $x(t) \to 0$ exponentially as $t \to \infty$ but still have a periodic component.

    For this to occur, we need the characteristic polynomial to have solutions with real and imaginary parts. This occurs when the discriminant $b^2 - 4ac < 0$, and thus we need $b^2 - 4(2) < 0 \implies b^2 < 8$. This occurs, for instance, if $b = 2$. (Note that for $b < 0$, the solution would not be damped...as you will see below, the solutions would be *negatively damped* and would GROW to infinity.)

(b) Consider the second order, constant coefficient differential equation in $x(t)$:

$$x'' - 4x' + 4x = g(t). \tag{2}$$

i. Let $g(t) = 2e^{2t}$. What is the general solution to (2) in this (inhomogeneous) situation?
The characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2$. Thus we have one real, repeated root: $r = 2$. The general solution is

$$x_g(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

To find the particular solution, we use the method of undetermined coefficients and we guess that the solution has the form At^2e^{2t}. Plugging this in, we note that ..., and thus we conclude that $A = ...$, and the particular solution is $x_p(t) = 2e^{2t}$. Thus the general solution to the inhomogeneous problem is $x(t) = x_p(t) + x_g(t) = 2e^{2t} + c_1e^{2t} + c_2te^{2t}$.

ii. Let $g(t) = 0$ and $x(0) = 1,\ x'(0) = 0$.
This is just the homogeneous case, and thus we'll work with $x_g(t) = c_1e^{2t} + c_2te^{2t}$. We need to compute x_g' to use the second initial condition:

$$x_g'(t) = 2c_1e^{2t} + c_2e^{2t} + 2c_2te^{2t} = (2c_1 + c_2)e^{2t} + 2c_2te^{2t}.$$

We impose the initial conditions now: first, $1 = x_g(0) = c_1$; secondly, $0 = x_g'(0) = 2c_1 + c_2$. With $c_1 = 1$, we conclude that $c_2 = -2$. Thus the solution to the IVP is $x(t) = e^{2t} - 2te^{2t}$.
In this (homogeneous) case what is the solution satisfying (2) and the given *initial conditions*?

(c) Using the forced ODE

$$x'' + 9x = \sin(\omega t),$$

explain in about three sentences the *resonance* phenomenon.

The fundamental set here is purely sinusoidal, and the general solution to the homogeneous equation is $x_g(t) = c_1 \sin(3t) + c_2 \cos(3t)$. If we include a periodic forcing, the particular solution to the inhomogeneous problem will include a term of the form $x_p(t) = A\sin(\omega t) + B\cos(\omega t)$. However, if $\omega = 3$, which is to say that the periodic forcing matches the natural frequency of the system, the particular solution needs to be adjusted (as in the previous problem). In this case, we would have $x_p(t) = t[A\sin(3t) + B\cos(3t)]$. Note that the solution will remain bounded in the case where $\omega \neq 3$, but when $\omega = 3$, the solution grows unboundedly in time. This is resonance—when a solution is forced near its natural frequency, the solutions grow unboundedly in time.