August 11, 2025

Homework 1: Review Material

MATH 404, Fall 2025

Due: W, 9/3 in class

This assignment is meant to illustrate the mathematical ideas, notation, and tools that will be used in this class. It isn't intended to be particularly difficult—**though it will be time consuming**. It should help you refresh on concepts that may have become a bit fuzzy. *It is likely necessary, and encouraged, that you look up definitions or formulae to assist you.*

Please write up individual solutions, showing details. Where explanations are requested, clear and complete sentences are expected. Scratchwork will not be graded. Typesetting solutions would be best (see the syllabus). One should complete *most* of *most of* the questions for full points.

1. Integration I.

   (a) Assuming $x > 0$, compute, showing all work $\int_0^\infty (t-3)e^{-tx}dt$

   (b) Suppose that f, g are smooth functions that have the property that $f(0) = f'(0) = g(0) = g'(0) = 0$. Show that

   $$\int_0^L f^{(4)}(x)g(x)dx = \int_0^L f''(x)g''(x)dx + f'''(L)g(L) - f''(L)g'(L).$$

   (c) Showing your work, compute the following integrals for all integers $m, n \in \{1, 2, 3\}$:

   $$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx$$

   Present your answer as a 3×3 matrix/array.

   (d) Make a conjecture (two sentences) about $\int_{-\pi}^{\pi} f(x)g(x)dx$ when $f, g \in \left\{\frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$.

2. Series.

   For each of the following series, determine: if it converges **C**, or if it diverges **D**. Then state in a sentence or two how you arrived at your conclusion.

   (a) $\sum_{n=1}^{\infty} \pi^{-n}$ (b) $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$ (c) $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$

3. Vector Operations.

   (a) The divergence operator is typically written as $\text{div}(.) = \nabla \cdot (.)$. Given a vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ with components $\mathbf{F} = \langle f, g \rangle$, the 2D divergence in Cartesian coordinates is given by

   $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \nabla \cdot \langle f, g \rangle = \partial_{x_1} f + \partial_{x_2} g.$$

   In two sentences or so, explain what the divergence at a point $(\nabla \cdot \mathbf{F})(\mathbf{x})$ measures about the vector field $\mathbf{F}$ at the point $\mathbf{x}$?

   (b) Derive the product rule for the 2D divergence of a scalar-vector product: $\text{div } (u\mathbf{F}) = ...$ where u is a scalar function and $\mathbf{F} = \langle f, g \rangle$ (you may assume all functions are smooth).

4. Kernels of Some Operations.

   (a) Let $\mathbf{F}$ be a (smooth) conservative vector field with potential function ϕ. What must be true of curl $\mathbf{F} = \nabla \times \mathbf{F} = \nabla \times \nabla\phi$? *In other words, the curl of a conservative vector field is...*(finish the sentence and justify with a sentence or two.)

   (b) Let $\mathbf{C} = \nabla \times \mathbf{F}$ (C is a curl field). What must be true of div $\mathbf{C} = \nabla \cdot \mathbf{C} = \nabla \cdot (\nabla \times \mathbf{F})$? *In other words, the divergence of a curl field must be...*(finish and justify with a sentence or two.)

5. Integration, II.

   (a) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose Γ is "nice" curve in the plane. Consider the line integral

   $$\int_\Gamma f(\mathbf{x})ds.$$

   What does this line integral represent? (Be descriptive in a sentence or two.) How is it computed in practice? (What is the procedure if Γ and f are actually given?)

   (b) Compute the surface area (showing all work) of a spherical cap with height h cut from a sphere with radius R.

6. Integration, III.

   Recall that for any smooth, oriented curve $\Gamma \subset \mathbb{R}^2$ we can define the unit radial (with positive orientation) vector $\mathbf{r}(\mathbf{x}, \mathbf{y})$ and the unit outward normal vector $\mathbf{n}(x, y)$. For all values along Γ, we have $d\mathbf{r} \cdot d\mathbf{n} = 0$, where $d\mathbf{r} = \langle dx, dy \rangle$ and $d\mathbf{n} = \langle dy, -dx \rangle$.

   Consider C_1 to be the boundary of the semicircle of radius two in the upper half plane with standard orientation (counter clockwise). Compute the following in any way you'd like.

   (a) The average value of the function $f(x, y) = x + y + 2$ over the curve C_1.

   (b) The flux through C_1 of the vector field $\mathbf{F}(x, y) = \langle x^2y\ ,\ (1/3)x^3 + y \rangle$

   (c) Now consider C_2 to be the line segment going from $(-2, 0)$ to $(2, 0)$. Compute the circulation of $\mathbf{F}$ (as given above) over the closed curve $C = C_1 \cup C_2$.

7. The Laplacian.

   (a) Recall that for $f(x, y)$ we have $\Delta f = \nabla \cdot \nabla f$. In Cartesian coordinates, $\Delta f = f_{xx} + f_{yy}$. Consider polar coordinates with the change of variable mapping:

   $$\begin{aligned} \theta(x, y) &= \arctan(y/x) \\ r(x, y) &= \left[x^2 + y^2\right]^{1/2}. \end{aligned}$$

   Thinking of a f as $f(r, \theta)$, and using the *chain rule*, compute the expression for Δf in terms of r and θ—show all work.

   (b) Let Ω be a region bounded by a curve Γ, where Γ is a positively oriented, p.w. smooth, simple, closed curve in $\mathbb{R}^2$. (These are the hypotheses for *Green's Theorem.*) Recall that function u is called *harmonic* if $\Delta u = 0$.

      i. Argue that, if u is harmonic, then $\oint_\Gamma \nabla u \cdot \mathbf{n}\ ds = 0$ (line integral over Γ is zero).

      ii. Argue that, if u is harmonic and $u(x, y) = 0$ on Γ, then $\iint_\Omega \nabla u \cdot \nabla u\ dA = 0$. What can you infer about u in this case?

8. First order ODE.

   (a) Verify that $y(x) = x[1 + \cos(x)]$ solves the IVP:

   $$\frac{dy}{dx} = \frac{y}{x} - x\sin(x), \quad y(\pi) = 0.$$

   (b) Solve the ODE: $t^3 y' + 4t^2 y = e^{-t}, \quad t > 0.$

9. Modeling.

   Let $P(t) \geq 0$ be some population at time $t \geq 0$. Let $P(0) = P_0 \geq 0$. Consider the population model

   $$\frac{dP}{dt} = 2P(100 - P)(P - 20), \quad P(0) = P_0.$$

   $K = 100$ is the *carrying capacity* and $M = 20$ is the *healthy population minimum.*

   Describe the properties of the solution in about 3 sentences. (What are the equilibrium solutions? What are the ultimate possible outcomes for the population (as $t \to \infty$)? How does the "shape" of a solution and its ultimate outcome depend on the initial data P_0?) Use your intuition, and see what the DE tells you. Think about what the IVP is *trying* to model.

10. Second order constant coefficient ODE.

    (a) Consider the second order, constant coefficient differential equation in $x(t)$:

    $$x'' + bx' + 2x = 0. \tag{1}$$

    Find a number b so that solutions are damped, but not *overdamped*—i.e., such that solutions $x(t) \to 0$ as $t \to \infty$ but still have a finite period associated to them.

    (b) Consider the second order, constant coefficient differential equation in $x(t)$:

    $$x'' - 4x' + 4x = g(t). \tag{2}$$

    i. Let $g(t) = 2e^{2t}$. What is the *general solution* to (2)?

    ii. Let $g(t) = 0$ and $x(0) = 1,\ x'(0) = 0$. Give the solution to the *initial value problem.* How do you know this is on the only solution? (One sentence.)

    (c) Consider the forced ODE

    $$x'' + 9x = \sin(\omega t).$$

    Find a value of ω such that *resonance* occurs, and explain what this means.