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Fourier Practice Questions

MATH 404, Spring 2023

1. For this problem, let $V = \mathbb{R}^3$ be the standard three-dimensional Cartesian vector space with basis vectors $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$. Recall the inner-product in $\mathbb{R}^3$ is the standard dot product:

$$\mathbf{v} \cdot \mathbf{w} = \sum_{n=1}^{3} v_n w_n,$$

where $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$.

   (a) What is $\mathbf{i} \cdot \mathbf{j}$? What is $\mathbf{i} \cdot \mathbf{k}$? What is $\mathbf{j} \cdot \mathbf{k}$? What are $\mathbf{i} \cdot \mathbf{i}$, $\mathbf{j} \cdot \mathbf{j}$, $\mathbf{k} \cdot \mathbf{k}$?

   (b) What is $\mathbf{v} \cdot \mathbf{j}$? What is $\mathbf{v} \cdot \mathbf{j}$? What is $\mathbf{v} \cdot \mathbf{k}$?

   (c) Perform a "Fourier decomposition" of the vector $\mathbf{v}$; this is to say, what are the coefficients A_n in the sum:

$$\mathbf{v} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}?$$

2. Consider the vector space $V = L^2(0, L)$ with inner-product $(f, g) = \displaystyle\int_0^L f(x)g(x)dx$, and the set of Dirichlet eigenfunctions $\{s_n(x)\} = \left\{\sin\left(\frac{n\pi}{L}x\right)\right\}_{n=1}^{\infty}$.

   (a) Why is an eigenfunction $s_n(x) \in L^2(0, L)$? Explain.

   (b) What is (s_n, s_n)? What is (s_n, s_m) for $m \neq n$?

   (c) For some function $\phi \in L^2(0, L)$, the Fourier sine series is

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right).$$

   If the series converges, what *must* A_n be?

   (d) Thinking of $\phi(x)$ as *an expansion in eigenfunctions*, interpret (in about three sentences) this result in terms of the first problem.

3. Consider the function $\phi(x) = x\,(x - L)$.

   (a) What type of boundary conditions does the polynomial satisfy?

   (b) Determine the appropriate Fourier series by determining the associated coefficients A_n.

   (c) Repeat the previous parts for the function $\phi(x) = 3$.

   (d) Compute the Fourier coefficients (sine and cosine) for the function $\phi(x) = x$. What is different about this case?

4. Consider the general solution to the Dirichlet wave equation IBVP on $[0, L]$:

$$u(x, t) = \sum_{n=1}^{\infty} s_n(x)[A_n \sin(\lambda_n^{1/2} ct) + B_n \cos(\lambda_n^{1/2} ct)],$$

where $s_n(x)$ and λ_n are the appropriate eigenfunctions and eigenvalues.

Determine A_n and B_n so that the initial conditions $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$ (for $\phi, \psi \in L^2(0, L)$) are satisfied.