Fourier II

So the point is: given any $\varphi \in L^2(0,L)$ (like $\varphi(x) = x$ or $\varphi(x) = x^2$).

You can compute its Fourier series in terms of any set of eigenfunctions you like.

For compatibility, you want to make sure the Fourier series you choose corresponds to the BCs you have.

$$\begin{array}{ccc} & \underline{\text{BC of } \varphi(x)} & \\ & D, N, M, P, R & \\ \swarrow & & \searrow \\ \underline{\text{EVP}} & \longleftrightarrow & \underline{\text{F-series}} \\ D, N, M, P, R & & \sin, \cos, \text{half sine}, \\ & & \text{half cos}, \ldots \end{array}$$

Two examples

$\varphi(x) = 1$

F-cos series: $A_n = (\varphi, S_n)$ for

$$S_n = \sqrt{\tfrac{2}{L}} \cos\left(\tfrac{n\pi}{L} x\right) \qquad n = 0, 1, 2, 3, 4, \dots$$

$$A_n = (\varphi, S_n) = \begin{cases} \left(\frac{1}{\sqrt{L}}, 1\right) = \sqrt{L} & n = 0 \\ \left(\sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}\right), 1\right) = 0 & n = 1, 2, 3 \end{cases}$$

$$\varphi(x) = \sum_{n=0}^{\infty} A_n \cos\left(\tfrac{n\pi}{L} x\right) = A_0 S_0(x) + 0 + 0 + \dots$$

$$= \sqrt{L} \frac{1}{\sqrt{L}} = 1$$

F-sin series: $A_n = (\varphi, S_n)$ for

$$S_n = \sqrt{\tfrac{2}{L}} \sin\left(\tfrac{n\pi}{L} x\right) \qquad n = 1, 2, 3, \dots$$

$$A_n = \left(\sqrt{2/L} \sin\left(\tfrac{n\pi}{L} x\right), 1\right)$$

$$= \begin{cases} -\frac{\sqrt{2L}}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$\varphi(x) = 1 + \cos\left(\frac{7\pi}{L} x\right)$

Which series is appropriate?

~~sin~~ wrong BCs

cos ✓ What happens?

$$\varphi(x) = \sum_{n=\phi}^{\infty} A_n \cos\left(\frac{n\pi}{2} x\right) \cdot \sqrt{\frac{2}{L}} + A_0 \frac{1}{\sqrt{L}}$$

Where $A_n = \left(1 + \cos\left(\frac{7\pi}{2} x\right), S_n(x)\right)$

but orthogonality says:

$$A_n = 0 \quad \text{if} \quad n \neq 1 \text{ or } 7 \; !$$

so $\varphi(x) = A_7 \sqrt{\frac{2}{L}} \cos\left(\frac{7\pi}{2} x\right) + A_0 \frac{1}{\sqrt{L}}$

$A_7 = \left(\sqrt{\frac{2}{L}} \cos\left(\frac{7\pi}{2} x\right), \cos\left(\frac{7\pi}{2} x\right)\right)$

$\Rightarrow \varphi(x) = 1 + \cos\left(\frac{7\pi}{2} x\right)$

Ahh! THIS IS IMPORTANT.