Fourier (Fore-YAY!) I

Recall in calc II, you saw

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

$$\forall x \in \mathbb{R}$$

This is a power-series expansion of e^x. (Infinite polynomial.)

Fourier asked (motivated by the problems were solving): For what φ can I obtain: $x \in (0, L)$ $\quad x \overset{?}{=} 0, x \overset{?}{=} L$

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right)$$

or $$\varphi(x) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{L} x\right)$$

In what sense do we understand the equality above? Values at boundary?

Recall, for the Dirichlet HEQ,

we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2} x\right) \exp\left(-\left(\frac{n\pi}{2}\right)^2 Dt\right)$$

$\{A_n\}_{n=1}^{\infty}$ are free for us to choose.

We <u>need</u>: $u(x,0) \stackrel{!}{=} \varphi(x)$

So we need to <u>Force</u>

$$\varphi(x) \stackrel{?}{=} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right)$$

What should the A_n be...?

(For Neumann, $\varphi(x) \stackrel{?}{=} \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi}{L} x\right)$)

<u>Orthogonality</u>

From <u>HW 1</u> you showed:

$$\int_0^L \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases}$$

We write this in inner product notation:

$$\left(\sin\left(\frac{n\pi}{L}x\right), \sin\left(\frac{m\pi}{L}x\right)\right)_{L^2(0,L)} = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & m = n \end{cases}$$

$$\left(\cos\left(\frac{n\pi}{L}x\right), \cos\left(\frac{m\pi}{L}x\right)\right)_{L^2(0,L)} = \begin{cases} 0 & m \neq n \\ L & m = n = 0 \\ \frac{L}{2} & m = n \neq 0. \end{cases}$$

Let us rescale the eigenfunctions

Dirichlet, sine

$$S_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1, 2, 3, \ldots$$

Neumann, cosine

$$S_n = \begin{cases} \frac{1}{\sqrt{L}} & n = 0 \\ \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}x\right) & n = 1, 2, 3 \end{cases}$$

For EVPs, such S_n are orthonormal.

$(S_n, S_m) = 0$ when $n \neq m$, $= 1$ when $n = m$

This is crazy! special! rare! important!

Why? Suppose

$$\varphi(x) = \sum_{n=1}^{\infty} A_n s_n(x)$$

mult. by s_m + integrate

$$(\varphi(x), s_m) = A_1(s_1, s_m) + A_2(s_2, s_m) + A_3(s_3, s_m) + \dots + A_n(s_n, s_m) + \dots$$

$$= 0 + \dots + A_m(s_m, s_m) + 0 + \dots$$

$$= A_m \|s_m\|^2_{L^2(0,L)}$$

Rearrange:

Given

$$A_m = (\varphi, s_m)$$

known

So to make the series converge, we choose (for $\varphi(x)$ and given s_n)

$$A_n = (\varphi, s_n) \quad \forall\, n = 1, 2, 3, 4, \dots$$

Fourier coefficients

Theorem $\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$ or $\cos\left(\frac{n\pi}{L}x\right)\cdot\sqrt{\frac{2}{L}}$

The functions $\{S_n\}_{n=0,1,2,3\ldots}^{\infty}$ (as written before) form a basis for $L^2(0,L)$. This basis is orthonormal.

For any $\varphi \in L^2(0,L)$, we can make

$\varphi(x) = \sum A_n S_n$ converge in the sense of $L^2(0,L)$ by choosing

$A_n = (S_n, \varphi)$ for S_n an orthonormal set of eigenfunctions.

In general, any set of eigenfunctions can be made into an ONB on $L^2(0,L)$.

Next: Boundary conditions... and how do we use these facts...