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Fourier Series: Overview and Comments

MATH 404, Spring 2023

These notes are meant to be a short supplement for our discussions on Fourier series.

1 Separation of Variables to Fourier Series

In our discussions of separation of variables for second order equations (Heat, Wave, and Laplace), we arrived at the following separation problems:

$$u_t = Du_{xx} \qquad u_{tt} = c^2 u_{xx} \qquad u_{xx} + u_{yy} = 0 \tag{1}$$

$$u(x,t) = r(t)s(x) \qquad u(x,t) = r(t)s(x) \qquad u(x,y) = r(y)s(x) \tag{2}$$

$$\frac{r_t}{Dr} = \frac{s_{xx}}{s} = \lambda \qquad \frac{r_t}{c^2 r} = \frac{s_{xx}}{s} = \lambda \qquad -\frac{r_{yy}}{r} = \frac{s_{xx}}{s} = \lambda. \tag{3}$$

Note: energy arguments convinced us, in each case, that $\lambda \leq 0$, as long as we had homogeneous Neumann or homogeneous Dirichlet boundary conditions. (In fact, for most cases with nice boundary conditions (mixed homogeneous, or Robin with good signs) the eigenvalues will have strict sign conventions.

2 Eigenvalue Problem

In finishing the separation of variables argument, one first solves the *eigenvalue problem* (EVP). This corresponds to the spatial problem above in the function $r(x)$:

$$r_{xx}(x) - \lambda r(x) = 0, \quad \text{with imposed boundary conditions.}$$

The general solution to this ODE is:

$$r(x) = A\sin([-\lambda]^{1/2}x) + B\cos([-\lambda]^{1/2}x).$$

At this point, λ, A, and B are all unknowns.

When invoking the *homogeneous Dirichlet boundary conditions* on $x \in (0, L)$

$$s(0) = s(L) = 0,$$

we eliminate the cosine term with the BC at $x = 0$, and we obtain an infinite number of solutions $\lambda_n = -\left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3, ...$ from the BC at $x = L$. This yields the family

$$s_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right),$$

each of which is a spatial solution respecting the Dirichlet BCs.

Similarly, when invoking the *homogeneous Neumann boundary conditions* on $x \in (0, L)$

$$s_x(0) = s_x(L) = 0,$$

we eliminate the sine term with the BC at $x = 0$, and we obtain an infinite number of solutions $\lambda_n = -\left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3, ...$ from the BC at $x = L$. This yields the family

$$s_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right),$$

each of which is a spatial solution respecting the Neumann BCs. Note: for the Neumann case the zero eigenvalue $\lambda_0 = 0$ has a non-trivial eigenfunction: $s_0(x) = 1$.

3 Basic Fourier Series

Using the principle of superposition, we see that the following sums are solutions to the spatial eigenvalue problem respecting the appropriate boundary conditions:

Dirichlet $$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \qquad x \in (0, L) \tag{4}$$

Neumann $$\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right) \qquad x \in (0, L) \tag{5}$$

General $$\frac{b_0}{2} + \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{n\pi}{L}x\right) + b_n \cos\left(\frac{n\pi}{L}x\right)\right] \qquad x \in (-L, L). \tag{6}$$

The key question in solving the BVPs or IBVPs above is to recover the data from the series solution. This typically involves a given function $\phi(x) \in L^2(0, L)$ and the question: **When does a given Fourier series converge to $\phi(x)$?** This will involve:

- choosing the Fourier series we want to use,
- constructing the appropriate coefficients,
- and then determining convergence of the series (and in what sense).

Given $\phi(x)$, Using the fact that $\{\sin([-\lambda_n]^{1/2}x)\}_{n=1}^{\infty}$ and $\{\cos([-\lambda_n]^{1/2}x)\}_{n=0}^{\infty}$ are orthogonal families on $(0, L)$ (or $(-L, L)$), we obtain (using the "multiply and integrate" strategy) the following coefficient structure to recover $\phi(x)$:

$$a_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{2}{L}(\phi, s_{n,\sin}), \; n = 1, 2, 3, ...$$

$$b_n = \frac{2}{L}\int_0^L \phi(x) \cos\left(\frac{n\pi}{L}x\right) dx = \frac{2}{L}(\phi, s_{n,\cos}), \; n = 0, 1, 2, 3, ...$$

(When considering the full Fourier series on $(-L, L)$, one must adjust the size of the interval and then make the appropriate changes to the coefficient structure above.)

With these coefficients, if the Fourier series converges, it will converge to $\phi(x)$ (in some sense).

Since constructing the Fourier coefficients depends only on integration, given any $\phi(x) \in L^2(0, L)$, we can always construct the sequences $\{a_n\}$ and $\{b_n\}$. Convergence of the series back to $\phi(x)$ allows us to give the full series solution to our second order problem via separation of variables, superposition, and Fourier.

4 Extensions, Periodicity, and Convergence

One can note, immediately, that the Fourier sine and cosine series represent only one half of the period associated with these functions. Thus we can certainly consider $x \in (-L, 0)$ as well. As before, since the Fourier sine series respects Dirichlet boundary conditions, for a given function $\phi(x)$, the sine series (for $x \in (0, L)$) will actually converge to $\phi_o(x)$ (the odd extension) on $(-L, L)$. This is to say that the two series are equivalent (in some sense) below:

$$\phi(x) \sim \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right), \quad x \in (0, L) \quad \Longleftrightarrow \quad \phi_o(x) \sim \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right), \quad x \in (-L, L).$$

We have a similar situation for the Fourier cosine series and the even extension $\phi_e(x)$:

$$\phi(x) \sim \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right), \quad x \in (0, L) \quad \Longleftrightarrow \quad \phi_e(x) \sim \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right), \quad x \in (-L, L).$$

Now, with these extensions, it is clear that we have defined—for all three Fourier series, sine, cosine, and full—functions on $(0, L)$ (when the series converge). But, one can again note that these series (if they converge) should actually converge for all $x \in \mathbb{R}$, owing to the periodicity of the eigenfunctions. Thus, when they converge, they will converge to the **$2L$-periodic** extensions of $\phi_o(x)$, $\phi_e(x)$, or $\phi(x)$ (for the cases of sine, cosine, and full, respectively). The convergence results that we will quote below, then, can be stated for all $x \in \mathbb{R}$. If we wish to have results on $(0, L)$, $[0, L]$, $(-L, L)$, or $[-L, L]$, we simply invoke the convergence result on all of $\mathbb{R}$ for the appropriate $2L$-extension, then restrict our focus to the interval of interest.

For each of the following theorems, the convergence result holds for the $2L$-periodic extension.

Theorem 1. *If $\phi \in L^2$, then the Fourier series (sine, cosine, or full) converges in the L^2 sense. In particular, this means that we can say the Fourier series converges on $(0, L)$ or $(-L, L)$ (whichever is appropriate). If, in addition, ϕ is continuous on $[0, L]$ and ϕ' is piecewise continuous on $[0, L]$ (or $[-L, L]$, if that is appropriate), then the convergence of the Fourier series is point-wise for $x \in (0, L)$ (resp. $[-L, L]$).*

The above theorem does not address convergence at the boundary, which was an issue we discussed in class. If we impose strong conditions on ϕ, then we get a very strong notion of convergence of the Fourier series.

Theorem 2. *If ϕ itself satisfies the boundary conditions at $x = 0, L$ (or $x = -L$, depending on the domain of interest) and ϕ, ϕ', and ϕ'' exist and are continuous on $[0, L]$ (or $[-L, L]$), then the Fourier series (sine, cosine, or full) converges uniformly. This is a very strong notion of convergence.*

What is often common is that ϕ is nice...but not too nice. In particular, what happens if $\phi \in L^2$ but does not satisfy the boundary conditions associated to the given Fourier series? In this case we cannot hope to have the Fourier series (which must respect the boundary conditions) converge at $x = 0, L$. The last convergence theorem addresses this case. Again, think of ϕ as being the $2L$-periodic extension.

Theorem 3. *If ϕ, ϕ' are piecewise continuous on $[0, L]$ (or $[-L, L]$) then the Fourier series (sine, cosine, or full) converges for all $x \in \mathbb{R}$, and has the value*

$$\frac{1}{2}[\phi(x^+) + \phi(x^-)].$$

Thus, if x is a point of continuity for $\phi(x)$, then the series converges to $\phi(x)$. If x is a point of jump discontinuity, then the Fourier series converges at that point to the half of the size of the jump.

5 Complex Form of Fourier

Recalling the DeMoivre formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

it is not a far cry to see the relationship between Fourier series involving sine and cosine, and Fourier series involving complex exponentials. It is a good exercise to check that $\left\{\exp\left(i\frac{n\pi}{L}x\right)\right\}_{n=-\infty}^{\infty}$ is an orthogonal family. (Don't forget the complex conjugate in the inner product.) To a given function $\phi(x)$ on $(-L, L)$, we can then associate the Fourier series:

$$\phi(x) \sim \sum_{n=-\infty}^{\infty} c_n \exp\left(-i\frac{n\pi}{L}x\right),$$

where we have coefficients $\{c_n\}_{n=-\infty}^{\infty}$ given by

$$c_n = \frac{1}{2L}\int_{-L}^{L} \phi(x)\exp\left(i\frac{n\pi}{L}x\right).$$

In this setup we can connect the theory of Fourier series on $(-L, L)$ to that of the Fourier transform on $\mathbb{R}$ by taking a limit as $L \to \infty$. Let us rewrite the coefficient equation:

$$c_n = \frac{1}{2\pi}\int_{-L}^{L} \phi(x)\exp\left(i\frac{n\pi}{L}x\right)\left[\frac{\pi}{L}\right].$$

Now, let $\xi = \dfrac{n\pi}{L}$ and note that as $L \to \infty$, $\pi/L \to d\xi$, thinking of ξ as the frequency variable, and the sum in n will become an integral in ξ.

Writing the Fourier relation, we have:

$$\phi(x) = \sum_{n=-\infty}^{\infty} e^{-i\xi x}\left[\frac{1}{2\pi}\int_{-L}^{L} e^{i\xi x}\phi(x)dx\right]\left[\frac{\pi}{L}\right].$$

Now let $L \to \infty$...

$$\phi(x) = \int_{-\infty}^{\infty} e^{-i\xi x}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\phi(x)dx\right]d\xi.$$

This is the Fourier inversion theorem!