Day 1: Intro to Ordinary Differential Equations (Slides Not Typical for This Course)

Math 225

2/1/22

Foundational Material

FUNCTIONS

$$f:\ \mathbb{R} \to \mathbb{R} \qquad f:\ \mathbb{R}^n \to \mathbb{R} \qquad \vec{f}:\ \mathbb{R} \to \mathbb{R}^m \qquad \vec{F}:\ \mathbb{R}^n \to \mathbb{R}^m$$

Domain, Co-Domain, Range

In/Dependent variables:

$$x(t), \quad y(x); \quad u(x,t); \quad G(x(t),t); \quad F(y'(t), y''(t), t)$$

Notions of *change*: $\dot{x}(t) = \dfrac{dx}{dt}$, $f'(x) = \dfrac{df}{dx} = \dfrac{dy}{dx}$; $\quad \partial_x y = \dfrac{\partial}{\partial x} y = \dfrac{\partial y}{\partial x} = D_x y$

Notions of *integration* and *anti-differentiation*:

$$\int_a^b f(x)dx, \quad \iint_\Omega g(x,y)dA, \quad \int f(x)dx, \quad \int_0^x f(s)ds = F(x)$$

Sequences, series, and associated notation: a_n, $\displaystyle\sum_i a_i$, $\displaystyle\sum_{n=1}^{\infty} \alpha_n x^n$

Required: Calculus I and II; Suggested: Calculus III and Linear Algebra

Modeling and DEs

Differential Equations—*partial* (PDEs) versus *ordinary* (ODEs)—are the language of physical sciences (*continuum* phenomena)

Mathematical modeling of phenomena:

Newton's second law, conservation of energy, Hooke's law, principle of virtual work, Hamilton's principle, Ficke's law, Fourier's law, etc.

ODEs and PDEs are branches of mathematics unto themselves

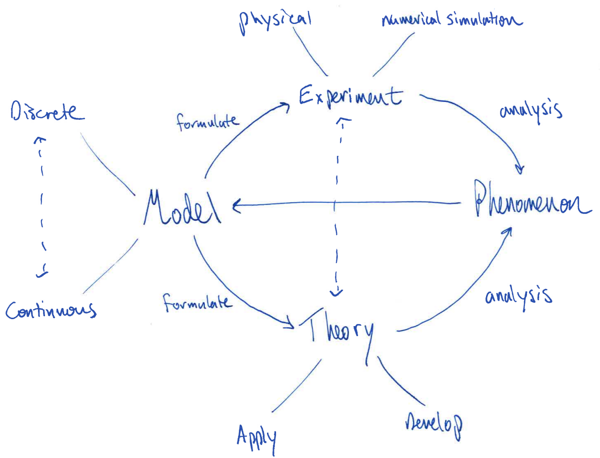
Discrete versus continuous

$$\int_a^b f(x)dx \quad \leftrightarrow \quad \sum_{i=1}^{N} f(x_i^*)\Delta x; \qquad \left.\frac{df}{dx}\right|_{x_0} \quad \leftrightarrow \quad \frac{f(x_0+h)-f(x_0)}{h}$$

Limiting procedures, approximations, THE NATURE OF THE UNIVERSE!

Applied Mathematics: ~~The~~ A Big Picture

+ *math motivated by a problem in the "real world"*
+ studying *phenomena of interest* using mathematical models
+ *developing new theory* or *applying existing theory* in doing so



Motivational ODE and PDE

MASS-SPRING SYSTEM (Damped Harmonic Oscillator):

$x(t)$ a real-valued function of time t:

$$mx''(t) + dx'(t) + kx(t) = 0$$

▸ In Vacuo Beam

ELASTIC BEAM: $w(x,t)$ is a real-valued function of x and t:

$$\partial_t^2 w(x,t) + k\partial_t w(x,t) + D\partial_x^4 w(x,t) = 0$$

Two notions of *change* in one equation: $\partial_t = \frac{\partial}{\partial t}, \quad \partial_x = \frac{\partial}{\partial x}$;

Evolution of states along t: $\left\{ \big(w(t,x), w_t(t,x)\big) \ : \ x \in (0,L) \right\}$

▸ In Vacuo

A Phenomenon of Interest: Flutter!

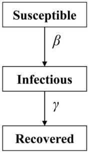
FLUTTER—A systemic instability in a flow-structure system occurring when the natural *modes* of the structure are *destabilized* by aerodynamic loading at the interface

Tacoma Narrows Bridge Catastrophe, 1940; “Galloping Gertie”

Longitudinal Torsional Collapse

A Phenomenon of Interest: Disease Dynamics (SIR)

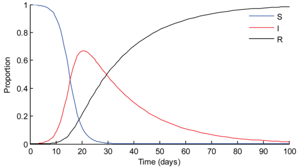
Succeptible/Infectious/Recovered Population Dynamics



$$\frac{dS}{dt} = -\beta S I$$

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$



Some Famous ODEs and Terminology

$P' = kP$ and $P' = kP(K-P)$ [Exponential and Logistic Growth]

$y' + y = y^2$ [Bernoulli's Equation]

$\ddot{x} + d\dot{x} + [k/m]x = F(t)$ and $\ddot{x} + \omega_0^2 x = 0$ [Oscillators]

$x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 4)y = 0$ [Bessel's Equation]

$\dfrac{d^2x}{dt^2} - \mu(1-x^2)\dfrac{dx}{dt} + x = 0$ [Van der Pol's Equations]

$\ddot{x} + f(t)x = g(t)$ and $y'' + \big(a - 2q\cos(2x)\big)y = 0$ [Hill/ Mathieu Equations]

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}$$ [Lorenz' Equations]

Differential Operators and Terminology

Operator L: $\quad L(x), \quad L(x, \frac{d}{dx}, \frac{d^2}{dx^2}), \quad L(x, \frac{d}{dx}, f), \ldots$

$L[f] = af'' + bf' + cf, \quad L(x)[f] = a(x)f'(x), \quad L(x, f) = x\sin(f(x))$

A ODE is **homogeneous** if all terms appearing depend on the solution variable.

$$x'' + \cos(2t)x = 0 \qquad \text{versus} \qquad x'' + \cos(2t)x = \sin(t).$$

Order: The highest order derivative present.

Coefficients: Things in front of the solution (dependent variable) and its derivatives.

Data: Boundary, Initial, Inhomogeneous terms