$u = e^{at} \cos(bx)$

$u(x,0) = \cos(bx)$

$u_t = a e^{at} \cos(bx)$

$u_{xx} = -e^{at} b^2 \cos(bx)$

$0 \stackrel{?}{=} u_t - D u_{xx} = a \cancel{e^{at} \cos(bx)} + D b^2 \cancel{e^{at} \cos(bx)}$

$a = -Db^2$

(The way you're supposed to do it.)

①

$$\begin{cases} u_t + D u_{xx} = 0 \\ u(x,0) = \cos(bx) \end{cases}$$

The way you should not do it.

Convolution solution:

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} \cos(by)\, e^{-\frac{(x-y)^2}{4Dt}}\, dy$$

Let $z = \frac{x-y}{2\sqrt{Dt}} \rightarrow y = x - 2\sqrt{Dt}\, z$

$$u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cos(b[x - 2\sqrt{Dt}\, z])\, e^{-z^2}\, dz$$

use (α, β) identity

$$u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \left(\cos(bx)\cos(2b\sqrt{Dt}\, z) + \sin(bx) \cdot \sin(2\sqrt{Dt}\, z) \right) dz$$

$\sin(cz)$ is odd

so $e^{-z^2} \sin(cz)$ is odd...

so

$$\int_{-\infty}^{\infty} e^{-z^2} \sin(n x)\sin(b2\sqrt{Dt}\,z)\,dz = 0$$

so

$$u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \cos(nx)\cos(b2\sqrt{Dt}\,z)\,dz$$

so we need to know

$$I = \int_{-\infty}^{\infty} \cos(cz)\, e^{-z^2}\,dz \qquad \text{so} \quad c = 2\sqrt{Dt}\cdot b$$

use $\cos(cz) = \dfrac{e^{icz} + e^{-icz}}{2}$ plug in

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{icz} e^{-z^2} + e^{-icz} e^{-z^2} \right) dz$$

Complete the square

$$I = \frac{1}{2}\int_{-\infty}^{\infty} e^{-(z-\frac{ic}{2})^2} e^{-\frac{c^2}{4}} + e^{-(z+\frac{ic}{2})^2} e^{-c^2/4}\,dz$$

$$I = \frac{e^{-c^2/4}}{2} \int_{-\infty}^{\infty} e^{-\left(z - \frac{ic}{2}\right)^2} + e^{-\left(z + \frac{ic}{2}\right)^2} \, dz$$

use a "fake" u-sub (twice:

$$u = z - \frac{ic}{2}, \quad u = z + \frac{ic}{2}$$

To get gaussian integrals.

$$I = \frac{e^{-c^2/4}}{2} \left[\sqrt{\pi} + \sqrt{\pi} \right] = \sqrt{\pi}\, e^{-c^2/4}$$

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so

$$u = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} \cos(bx) \cos(2\sqrt{Dt}\, z) \, dz$$

$$= \frac{\cos(bx)}{\cancel{\sqrt{\pi}}} \left[\cancel{\sqrt{\pi}}\, e^{-[2\sqrt{Dt}]^2/4} \right]$$

$$= e^{-Dtb^2} \cos(bx)$$