Putting It all together once more ①

Fully solve IBVP for Neumann Heat on $\Omega = (0,L)$

$$\begin{cases} u_t - D u_{xx} = 0 & x \in (0,L),\ t > 0 \\ u_x(0,t) = u_x(L,t) = 0 & t \geq 0 \\ u(x,0) = \varphi(x) & x \in [0,L],\ t = 0 \end{cases}$$

Separation yields: $\begin{cases} T_{n,t} = \lambda_n D T_n \\ S_{n,xx} = \lambda_n S_n \end{cases}$

So $T_n(t) = \exp(\lambda_n D t)$

EVP is Neumann, so:

$$S_n(x) = \begin{cases} \frac{1}{\sqrt{L}} & n = 0 \\ \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L} x\right) & n = 1, 2, 3, \dots \end{cases}$$

Hence $u_n(x,t) = \exp\left(-D\left(\frac{\lambda n}{\pi}\right)^2 t\right) S_n(x)$

Superpose:

$$u(x,t) = A_0 S_0 + \sum_{n=1}^{\infty} A_n \sqrt{2/L} \cos\left(\frac{n\pi}{L} x\right) e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

u solves the PDE by construction if the Fourier series converges.

The Fourier (cosine) series converges so long as $\varphi \in L^2(0,L)$. And if φ' is continuous s.t. $\varphi'(0) = \varphi'(L) = 0$, then $u_x(0,t) = u_x(L,t) = 0$.

How to get IC?

$$u(x,0) = A_0 S_0 + \sum_{n=1}^{\infty} A_n S_n$$

so for $S_n(x) = \begin{cases} 1/\sqrt{L} & n=0 \\ \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}x\right) & n = 1,2,3,\ldots \end{cases}$

Let $A_n = (S_n, \varphi)$.

With this choice, $u(x,t)$ solves the IBVP.

We will discuss Qualitative properties soon.

Dirichlet Wave EQ^n

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in (0,L),\ t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = \varphi(x);\ u_t(x,0) = \psi(x) & x \in [0,L] \end{cases}$$

Separation yields: $\begin{cases} T_{n,tt} = c^2 \lambda_n T_n \\ S_{n,xx} = \lambda_n S_n \end{cases}$

EVP is Dirichlet

$S_n(x) = \sin\left(\frac{n\pi}{L} x\right) \cdot \sqrt{\frac{2}{L}}$ $\quad n = 1, 2, 3, \ldots$

$T_n(t) = c_n \cos\left(\frac{cn\pi}{L} t\right) + d_n \sin\left(\frac{cn\pi}{L} t\right)$

So reconstruct, superpose:

$$u(x,t) = \sum_{n=1}^{\infty} S_n(x)\left(A_n \sin\left(\frac{cn\pi}{L} t\right) + B_n \cos\left(\frac{n\pi c}{L} t\right)\right)$$

As before: PDE on $(0,L) \times (0,\infty)$ ✓
BC on $\{0,L\} \times [0,\infty)$ ✓

IC: let $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} B_n S_n(x) \overset{!}{=} \varphi(x) \quad \text{when}$$

$B_n = (\varphi, S_n) \longleftarrow$ The Fourier sine coeffs.

For

$$u_t(x,t) = \sum_{n=1}^{\infty} S_n(x)\left(\frac{cn\pi}{L}\right)\left[A_n \cos\left(\frac{n\pi}{L}c\,t\right) - B_n \sin\left(\frac{n\pi c}{L}t\right)\right]$$

$$\psi(x) \overset{!}{=} u_t(x,0) = \sum_{n=1}^{\infty} A_n\left(\frac{cn\pi}{L}\right) S_n(x)$$

Choose Fourier sine:

$$A_n \cdot \frac{cn\pi}{L} \overset{!}{=} (\psi, S_n)$$

so choose $A_n = \dfrac{L}{cn\pi}(\psi, S_n)$ ← all known for $\psi \in L^2$

$S_n(x)$ called <u>harmonies</u> associated to fundamentals $\lambda_n^{1/2} c$ ← frequencies

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \sin\left(\frac{n\pi c}{L}t\right) + B_n \cos\left(\frac{n\pi c}{L}t\right)\right]$$

↗ string oscillations