A second order linear equation in two variables $x_1$ + $x_2$

Solution variable $u(x_1, x_2)$

equation

(*) $A\partial_{x_1}^2 u + 2B\partial_{x_1}\partial_{x_2} u + C\partial_{x_2}^2 u$

$+$ [lower order linear terms] $= 0$

Note: 1) the lower order terms don't affect this classification

2) DON'T FORGET THE "2" IN FRONT OF B

3) A, B, C can be functions of $x_1$ + $x_2$. The equation is still linear. If, for instance, $A = A(u)$ or $A(u_x)$ the equation would no longer be linear.

Classification In any $(x_1, x_2)$ region where:

$B^2 - AC > 0$, (*) is hyperbolic

$B^2 - AC < 0$, (*) is elliptic

$B^2 - AC = 0$, (*) is parabolic

## Classic Examples

(1) Wave equation in 1-space dimension

$$u_{tt} - c^2 u_{xx} = 0$$

$x_1 = t$ + $x_2 = x$

$A = 1 \quad B = 0 \quad C = -c^2$

$B^2 - AC = 0 - (1)(-c^2) = c^2 > 0$

The wave equation is hyperbolic

(2) Laplace's equation in $\mathbb{R}^2$

$$\Delta u = 0 \implies u_{xx} + u_{yy} = 0$$

$x_1 = x, \quad x_2 = y$

$A = 1 \quad B = 0 \quad C = 1$

$B^2 - AC = 0 - (1)(1) = -1 < 0$

Laplace's equation is elliptic

(3) Heat equation in 1-space dimension

$$u_t - D u_{xx} = 0 \qquad D = \text{const}$$

lower order term; so $AC = 0, \; B = 0$

$B^2 - AC = 0 \implies$ The heat equation is parabolic.

# Tricomi's equation

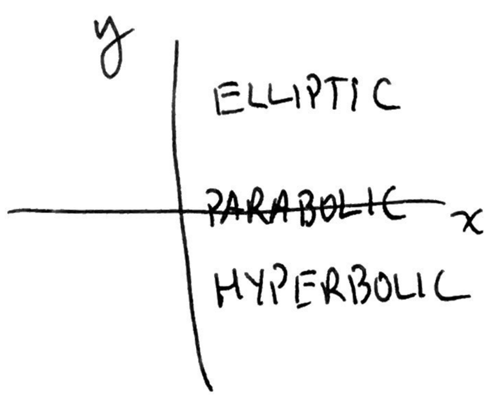
$$(**) \quad u_{yy} + y\, u_{xx} = 0$$

Linear second order equation with variable coefficient.

$A(x,y) = y \ , \ B = 0 \ , \ C = 1$

$B^2 - AC = -y$

So when: $\begin{cases} y > 0 & (**) \text{ is elliptic} \\ y = 0 & (**) \text{ is parabolic} \\ y < 0 & (**) \text{ is hyperbolic} \end{cases}$




The equation type changes in $(x,y)$ plane since the equation itself changes in the plane via the variable coefficients.

SEE HW 2 # 13 (bonus)
for more discussion about where these come from.

## An "example"

$$L = (2y)\partial_x^2 + (x+y)\partial_x\partial_y + (2x)\partial_y^2 \qquad (***)$$

$$A = 2y \qquad B = \frac{x+y}{2} \qquad C = 2x$$

$$B^2 - AC = \frac{(x+y)^2}{4} - 4xy$$

set $= 0$ ; $L$ will be parabolic

$$0 = B^2 - AC = \frac{(x+y)^2}{4} - 4xy$$

$$(x+y)^2 - 16xy = 0$$

$$x^2 + y^2 - 14xy = 0$$

Complete the square in $x$:

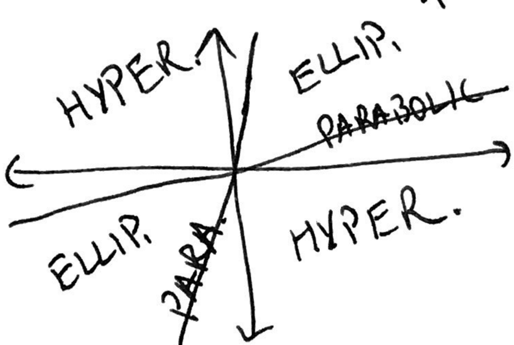
$$(x-7y)^2 - 49y^2 + y^2 = 0$$

so

$$(x-7y)^2 = 48y^2 \quad \text{take sqrt (don't neglect } \pm)$$

yields

$$x = (7 \pm 4\sqrt{3})y \qquad \underline{\text{two lines}}$$




Get the other regions by testing pts $(1,1)$, $(-1,1)$, $(1,-1)$, $(-1,-1)$ in $(x+y)^2 - 16xy$ ($>$ or $<$) $0$.