Lecture 1: Intro to PDE

Math 404

8/28/19
FUNCTIONS

\[ f : \mathbb{R} \rightarrow \mathbb{R} \quad f : \mathbb{R}^n \rightarrow \mathbb{R} \quad \vec{f} : \mathbb{R} \rightarrow \mathbb{R}^m \quad \vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^m \]

Examples?

Vector notation: \( \vec{v}, \ v_i, \ \vec{v} \cdot \vec{w}, \ v_i v_i \)

Notions of change: \( \partial_i, \nabla, D_{\vec{v}}, \text{div}, \text{curl}, \Delta \)

Notions of integration: \( \int_a^b dx, \iiint_{\Omega} dA, \iiint_{\Omega} dV, \oint ds, \iint dS \)

Sequences, series, and associated notation: \( a_n, f_n(x), \sum_i a_i, \sum_{n=1}^{\infty} f_n(x) \)

Calculus II and III, ODE

HW1—[http://webster.math.umbc.edu/HW1.pdf](http://webster.math.umbc.edu/HW1.pdf)
GET STARTED!
Partial Differential Equations (versus *ordinary* differential equations) are the language of physical sciences (*continuum* phenomena)

Mathematical modeling of phenomena:
Newton’s second law, conservation of energy, principle of virtual work, Hamilton’s principle, principle of least action

A branch of mathematics unto itself

PDEs do arise in application to other pure math fields; \( L(x, y, u, D^\alpha) = 0 \)

Discrete versus continuous

\[
\int_a^b f(x) \, dx \leftrightarrow \sum_{i=1}^N f(x_i^*) \Delta x; \quad \left. \frac{df}{dx} \right|_{x_0} \leftrightarrow \frac{f(x_0 + h) - f(x_0)}{h}
\]

Limiting procedures
Motivational Examples

$x(t)$ a real-valued function

\[ mx''(t) + dx'(t) + kx(t) = 0 \]

$u(x, t)$ is a real valued function

\[ u_{tt}(x, t) + du_t(x, t) - c^2 u_{xx}(x, t) = 0 \]

Different notions of change in the same equation; evolution of states along one variable
**Some Famous PDEs and Some Terminology**

\[ z(x, t) \text{ a real-valued function} \]
\[ z_t + a(x, t)z_x = f(x, t) \]

\[ \theta(\vec{x}, t) \text{ a real-valued function} \]
\[ \theta_t = K(\vec{x}) \Delta \theta \]

\[ \Psi(\vec{x}, t) \text{ a complex-valued function} \]
\[ \Psi_t = i\hbar \Delta \Psi + V(\Psi) \]

\[ \vec{v}(\vec{x}, t) \text{ a vector-valued function, } p(\vec{x}, t) \text{ a real-valued function} \]
\[ \vec{v}_t - \nu \Delta \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = 0 \]

\[ w(x, t) \text{ a real-valued function} \]
\[ w_{tt} + Dw_{xxxx} = f(w) \]

\[ u(\vec{x}, t) \text{ a real-valued function} \]
\[ \Delta u = F \]
Domains and Boundaries I

Domain $\Omega \subset \mathbb{R}^n$

$\Gamma = \partial \Omega$

$\vec{n}(\vec{x}) = \nu(\vec{x})$ in 2 or 3-D.

(picture)

What about tangential? What about boundary regularity? (2-D? 3-D?)

Examples:

$\mathbb{R}, \ [0, \infty), \ [a, b], \ [a, b] \times [a, b], \ \text{blobs}$

Open versus closed sets, closure
PDEs and boundary conditions, example:

\[
\begin{align*}
\Delta u(x, y) &= -\lambda u \quad \text{in } \Omega \\
\n u &= 0 \quad \text{on } \Gamma \\
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
 u_{xx} + u_{yy} + \epsilon u &= F(x, y) \quad \text{in } \Omega \\
 \nabla u \cdot \vec{n} &= 0 \quad \text{on } \Gamma 
\end{cases}
\end{align*}
\]

\(C^\alpha(\Omega)\) spaces; example: \(u \in C^2(\Omega) \cap C^0(\overline{\Omega})\)
Differential Operators and Terms I

General notation: \( \partial_i, \partial_{x_i}, \partial_x, D, D^\alpha \)

Operator \( L \): \( L(x), L(x, \partial_x), L(x, \partial_x, u), \ldots \)

\[
L[f] = af'' + bf' + cf, \quad L(x)[f] = a(x)f'(x), \\
L(f) = a(x)f^2(x), \quad L(x, f) = x \sin(f(x)) \\
Af = -\Delta f, \quad A(\vec{x})f = -\sum_{i,j}^n a_{ij}(\vec{x})\partial_{x_i}\partial_{x_j}f(x_1, \ldots, x_n).
\]

A PDE/BVP is **homogeneous** if all terms depend on the solution variable.

**Order**: The highest order derivative present; sometimes we talk about spatial order versus temporal order.

**Coefficients**: Things in front of differential operators.

**Data**: Boundary, Initial, Inhomogeneous terms
\[ \partial_x u = 0 \]
\[ \partial_x u + \partial_t u = 0 \]
\[ \partial_x u + \partial_t u = f(x) \]
\[ a(x)\partial_x u + \partial_t u = f(x, t) \]
\[ \partial_x u + \partial_t u = f(u) \]
\[ g(u)\partial_x u + \partial_t u = f(x) \]
\[ G(\partial_x u) + \partial_t u = 0 \]