Heat Equation Activity Solutions

MATH 404, Fall 2019

This activity is worth 10 points total (in-class points = quiz/activity points). Use separate sheets of paper. Try to work orderly, and turn in what you have at the end of class.

1. **Initial conditions.** Consider the “convolution with the heat kernel solution”:

   \[ u(x,t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} \phi(y) \exp \left( -\frac{(x-y)^2}{4Dt} \right) dy. \]

   We want to investigate the initial condition here by taking \( t \downarrow 0 \) in a particular way.

   (a) Consider the variable \( r = \frac{x-y}{2\sqrt{Dt}} \). Compute \( dr \) in terms of \( dy \), and solve for \( y \) in the expression for \( r \).

   \[ dr = -\frac{1}{2\sqrt{Dt}} \, dy, \quad y = x - (2\sqrt{Dt})r. \]

   (b) Substitute these values appropriately into the integral expression for \( u(x,t) \). Simplify.

   \[ u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \phi \left( x - (2\sqrt{Dt})r \right) e^{-r^2} \, dr. \]

   (c) In the result from the previous part, compute \( \lim_{t \downarrow 0} u(x,t) \). (You should get \( f(x) \) as your final answer.)

   \[ \lim_{t \downarrow 0} u(x,t) = \frac{\phi(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} \, dr = f(x). \]

   (d) Thus, given the heat equation \( u_t = Du_{xx} \) with initial condition \( u(x,0) = f(x) \), write out the solution to the Cauchy problem.

   We know that \( \int_{-\infty}^{\infty} f(y)S(x-y,t)dy \) solves the heat equation, and we have just shown that

   \[ \lim_{t \downarrow 0} \int_{-\infty}^{\infty} f(y)S(x-y,t)dy = f(x). \]

   Thus

   \[ u(x,t) = \int_{-\infty}^{\infty} f(y)S(x-y,t)dy \]

   is the solution to the heat equation Cauchy problem (as long as we interpret the initial condition in terms of this limit).

2. **Energies.** Define \( E(t) = \frac{1}{2} \int_{-\infty}^{\infty} [u(\xi,t)]^2 d\xi \).

   (a) Multiply the heat equation by \( u(x,t) \) and then integrate for all \( x \in \mathbb{R} \); then integrate by parts in space in order to obtain an *energy relation* (using the expression for \( E(t) \) above).

   Multiplying and integrating give us

   \[ \int_{-\infty}^{\infty} u_t(\xi,t)u(\xi,t)d\xi - D \int_{-\infty}^{\infty} u_{xx}(\xi,t)u(\xi,t)d\xi = 0. \]

   Using the relation \( \frac{d}{dt} [f^2] = 2f'(t)f(t) \) and integrating by parts in \( x \) in the second term, we have

   \[ \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} [u(\xi,t)]^2 d\xi + D \int_{-\infty}^{\infty} [u_x(\xi,t)]^2 d\xi - D[u_x(\xi,t)u(\xi,t)]_{x \to \infty} = 0 \]
Assuming that the solution $u(x, t)$ decays to zero as $|x| \to \infty$, the boundary term goes away, yielding

$$
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} [u(\xi, t)]^2 d\xi + D \int_{-\infty}^{\infty} [u_x(\xi, t)]^2 d\xi = 0.
$$

Then, we can use the FTC to rewrite:

$$
\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} [u(\xi, t)]^2 d\xi = -D \int_{-\infty}^{\infty} [u_x(\xi, t)]^2 d\xi \quad \iff \quad E(t) = E(0) - D \int_0^t \int_{-\infty}^{\infty} [u_x]^2 d\xi d\tau.
$$

(b) Note, from your previous answer, that energies decay. Explain.

From the first description above, we see that the time rate of change of $E(t)$ is negative, indicating decay. From the second description, it is clear that since $[u_x]^2$ is a positive function, its integral is also positive and being subtracted from $E(0)$, so $E(t) \leq E(0)$. In fact, for any interval of time $[s, t]$, we will have $E(t) \leq E(s)$, which is another way of saying that $E(t)$ is a decreasing function.