November 18, 2019

Homework 6: The Boundary and Separation of Variables I

MATH 404, Fall 2019

This assignment is due: M, 12/2 in class.

This assignment is about boundary value problems, and separation of variables. It is long; don’t put it off.

1. **Energy Methods.** For each IBVP below, we will use the energy relation to show uniqueness.

   (a) Consider the following IBVP (of Dirichlet type) for the heat equation for \( x \in [0, 1] \):

   \[
   \begin{cases}
   u_t = u_{xx} & x \in (0, 1), \ t > 0, \\
   u(x, 0) = \phi(x) & x \in [0, 1], \\
   u(0, t) = f(t); \ u(1, t) = g(t) & t \geq 0.
   \end{cases}
   \]

   Let \( u_1 \) be a solution with data \( \phi_1(x), f_1(t), g_1(t) \) and \( u_2 \) be a solution with data \( \phi_2(x), f_2(t), g_2(t) \).

   i. Let \( z = u_1 - u_2 \). Using the the IBVP that \( z \) solves, derive the energy relation for

   \[
   E_z(t) = \frac{1}{2} \| |z| \|_{L^2(0, 1)}^2.
   \]

   ii. Use this energy relation to deduce uniqueness of solutions in the sense of \( L^2(0, 1) \).

   iii. For \( f_i = g_i \equiv 0 \), explain how the energy relation also gives continuous dependence in the sense of \( L^2(0, 1) \). Write a clear sentence for each of these three points.

   (b) Consider the following IBVP (of Neumann type) for the wave equation for \( x \in [0, 1] \):

   \[
   \begin{cases}
   u_{tt} = u_{xx} & x \in (0, 1), \ t > 0, \\
   u(x, 0) = \phi(x); \ u_t(x, 0) & x \in [0, 1], \\
   u_x(0, t) = 0; \ u_x(1, t) = 0 & t \geq 0.
   \end{cases}
   \]

   Let \( u_1 \) be a solution with data \( \phi_1(x), \psi_1(x) \) and \( u_2 \) be a solution with data \( \phi_2(x), \psi_2(x) \).

   i. Let \( z = u_1 - u_2 \). Using the the IBVP that \( z \) solves, derive the energy relation for

   \[
   E_z(t) = \frac{1}{2} \left\{ \| z_t \|_{L^2(0, 1)}^2 + \| z_x \|_{L^2(0, 1)}^2 \right\}.
   \]

   ii. Use this energy relation to deduce: conservation of energy, uniqueness of solutions in the sense of \( L^2(0, 1) \), and continuous dependence in the sense of \( L^2(0, 1) \). Write a clear sentence for each of these three points.

2. (Strauss 2.3 #2) **Maximum Principle.** Consider a solution to the heat equation \( u_t = Du_{xx} \) for \( x \in [0, L] \) and \( t \in [0, \infty) \).

   (a) Let \( M(T) \) denote the maximum of \( u(x, t) \) in the rectangle \( \{0 \leq x \leq L\} \times \{0 \leq t < T\} \). Does \( M(T) \) increase or decrease as a function of increasing \( T \)? Explain.

   (b) Suppose that \( u_1(x, t) \) and \( u_2(x, t) \) are two solutions to the diffusion equation above. Suppose further that \( u_1 \leq u_2 \) for: \( x = 0, x = L, \) and \( t = 0 \). Why, then, is \( u_1(x, t) \leq u_2(x, t) \) for all \( t \in [0, \infty) \) and all \( x \in [0, L] \)? Explain in detail.
3. **Steady State.** Consider the *steady state* problem \( u_{xx} = 0 \) on \( x \in [0, L] \). Note that this is the steady state problem for both the heat and wave equation.

(a) Assume the solution \( u(x) \) is smooth. What can you deduce about it (using Calc. 1)? Explain how the boundary conditions at \( x = 0 \) and \( x = L \) are involved here.

(b) Now consider the inhomogeneous Dirichlet problem for the heat equation with \( x \in [0, L] \):

\[
\begin{cases}
   u_t - Du_{xx} = 0, & x \in (0, L) \\
   u(x, 0) = \phi(x) \\
   u(0, t) = g_1(t); \quad u(L, t) = g_2(t).
\end{cases}
\]

We will transform this problem into one with homogeneous boundary conditions.

i. Assuming that \( g_i(t) \) are given for all \( t \), compute a linear function \( L(x, t) \) which connects \((0, g_1(t))\) to \((L, g_2(t))\) for any value of \( t \).

ii. Consider a new variable \( w(x, t) = u(x, t) - L(x, t) \). Assuming \( u(x, t) \) is a solution, plug this into the PDE. What PDE does \( w \) solve?

iii. You achieved homogenous boundary conditions; what was the cost of doing this?

4. (Strauss 4.2 #1) **Eigenmusic.** Consider the Dirichlet wave equation with homogeneous boundary conditions as a model for a violin string.

(a) Use the series solution to explain (in a sentence or two) why “clamping” the string exactly at its midpoint raises the pitch by one octave.

(b) In a similar way, explain why the pitch increases as the string is tightened.

5. (Strauss 4.2, Bell 14) **Separation of Variables for Neumann.**

(a) Complete the full separation of variables argument (as we did in class) for the homogeneous Neumann heat equation:

\[
\begin{cases}
   u_t = Du_{xx} \\
   u_x(0, t) = u_x(L, t) = 0.
\end{cases}
\]

You should separate, solve the eigenvalue problem, and superpose. Note that we do not include initial conditions, and thus you should give your answer in the most general sense (incorporating the most general initial condition possible).

(b) What happens to the solution as \( t \to \infty \)?

(c) Without repeating all of the previous steps, what changes for the solution of the wave equation \( u_{tt} = c^2 u_{xx} \) (with homogeneous Neumann conditions)? Give your complete solution, as above, being careful not to do unnecessary work.

6. (Bell 14, #5) **Wave with Damping.** Consider the homogeneous Dirichlet-Neumann mixed wave equation with damping on \( x \in [0, 1] \):

\[
\begin{cases}
   u_{tt} - c^2 u_{xx} + du_t = 0 \\
   u(x, 0) = \phi(x); \quad u_t(x, 0) = 0 \\
   u(0, t) = u_x(1, t) = 0
\end{cases}
\]

Use separation of variables to obtain the series solution when \( d < 2\pi c \).
7. (Bonus) (Strauss 2.3 #8) **Robin Energy Relation.** Consider the heat equation on \( x \in [0, L] \) with the Robin conditions:

\[
\begin{align*}
    u_x(0, t) - a_0 u(0, t) &= 0 \\
    u_x(L, t) + a_L u(L, t) &= 0.
\end{align*}
\]

Recall that for the heat equation we have the energy

\[
E(t) = \frac{1}{2}(u, u)_L^2 = \frac{1}{2} \int_0^L (u(x, t))^2 \, dx.
\]

(a) Find an energy relation for this problem, following the steps from class.

(b) If \( a_0, a_L > 0 \), show that the boundary conditions are **dissipative**—namely that the rate of energy decrease in the problem is related to the values of \( a_0 \) and \( a_L \).

8. (Bonus) (Bell 15, Strauss 4.3) **The Robin Condition.** Consider the general Robin EVP.

\[
\begin{align*}
    s_{xx}(x) + \lambda s(x) &= 0 \\
    s_x(0, t) - a_0 s(0, t) &= 0 \\
    s_x(L, t) + a_L s(L, t) &= 0.
\end{align*}
\]

The values \( a_0, a_L \) (their sizes and magnitudes) have immense bearing on the eigenvalues. This problem is quite involved—see the reading. We will focus on simpler cases here.

(a) When \( a_0, a_L > 0 \), the boundary conditions are called **radiation conditions**. When \( a_0, a_L < 0 \) they are called **absorption conditions**. Explain this convention in a couple of sentences.

(b) Consider the simplified problem with \( L = 1 \), and \( a_L > 0 \):

\[
\begin{align*}
    s(0, t) = 0; \quad s_x(1, t) + a_L s(1, t) &= 0.
\end{align*}
\]

Show that \( \lambda > 0 \) for any solution.

(c) Derive the transcendental equation which characterizes the values of \( \lambda_n \). Show your work, and provide the approximate values of \( \lambda_i \) for \( i = 1, 2, 3 \).

(d) (Bonus) Consider the more general boundary conditions

\[
\begin{align*}
    s_x(0, t) - a_0 s(0, t) &= 0 \\
    s_x(L, t) + a_L s(L, t) &= 0.
\end{align*}
\]

Assume that \( a_0, a_L > 0 \). In this case, again, \( \lambda > 0 \). Derive the transcendental equation (showing work) that characterizes the eigenvalues \( \lambda_n \).

9. (Bonus) **Steady State, II.**

(a) Suppose that the heat flow in a metal rod \( x \in [0, 1] \) is governed by the following equations

\[
\begin{align*}
    u_t &= Du_{xx} + 1 \\
    u(0, t) &= 0, \quad u(1, t) = 1.
\end{align*}
\]

What will be the steady state temperature after a long time? Explain why it does not matter that no \( u(x, 0) = \phi(x) \) is given.

(b) Consider, instead, that the heat flow in a metal rod \( x \in [0, 1] \) is governed by

\[
\begin{align*}
    u_t &= Du_{xx} - hu \\
    u(0, t) &= u(1, t) = 1,
\end{align*}
\]

with \( h > 0 \). Find the steady state solution and describe what the heat flow is doing in the dynamics.
10. (Bonus) **Exponential Decay of the Heat Equation.** We will use a qualitative method observe that the heat equation decays exponentially to zero with homogeneous Dirichlet conditions.

(a) **Poincaré’s Inequality** Let \( f : [0, L] \rightarrow \mathbb{R} \) be a function such that \( f, f_x \in L^2(0, L) \) and \( f(0) = 0 \). Show that there is a constant \( C(L) \) such that:

\[
||f||^2_{L^2(0,L)} \leq C(L)||f_x||^2_{L^2(0,L)}.
\]

(b) Recall the (differential) energy identity for the solution \( u(x, t) \) to the heat equation with homogeneous Dirichlet boundary conditions:

\[
\frac{d}{dt} (||u||^2) = -2D||u_x||^2.
\]

Use part (a) to derive an energy *inequality*, written only in \( ||u|| \).

(c) Recall the \( \mu \)-method (integrating factor) from ODE for problems of the form:

\[
f' + \mu f \leq 0.
\]

Use this technique to obtain exponential decay of \( ||u(t)||^2_{L^2(0,L)} \).