Due: M, 9/23, in class

This assignment is about PDE basics and classifications.

1. Consider the heat equation \( u_t = u_{xx} \) in solution variable \( u(t,x) \), \( x \in \mathbb{R} \) and \( t \in (0, \infty) \).

   (a) (Bell, No #, p.3) Make the variable change \( z = \frac{x}{\sqrt{t}} \). Consider a solution \( u(x,t) \) of the form \( u(x,t) = f(z) \). Using the chain rule, plug in \( f(z) \) to the heat equation to arrive at an ODE in the new variable \( z \). Your answer should be an ODE (don’t solve it).

   (b) (Bell, 1.3) Verify, showing all steps, that:
       \[
       u(x,t) = \int_0^x \sqrt{t} e^{-s^2/4} \, ds
       \]
       satisfies the heat equation \( u_t = u_{xx} \).

2. (Bell, 1.4) Verify that a function \( u(x,y) = f(x)g(y) \) (for any smooth functions \( f, g : \mathbb{R} \to \mathbb{R} \)) is a solution to \( uu_{xy} = u_x u_y \).

3. (Bell, 2.2/DuZ 1.6.3) Indicate which of the following differential operators is nonlinear:
   
   (a) \( L[u] = u_x + x u_y \)
   
   (b) \( L[u] = u_x + u u_y \)
   
   (c) \( L[u] = \frac{\cos(y)}{(1 + x^2)^{1/2}} u_x + u_{xy} - \arctan(x/y) u \)
   
   (f) \( L[u] = \frac{u_{xx} + u_{yy}}{u} \)
   
   (g) \( L[u] = u_{xx} u_{yy} - 2 u_{xy}^2 \)

4. (Bell, 2.3) For each of the following equations, indicate whether or not it is linear. If it is linear, indicate whether or not it is homogeneous.
   
   (a) \( u_t - u_{xx} + 1 = 0 \)
   
   (c) \( iu_t - u_{xx} + \frac{u}{x} = 0 \)

   (b) \( u_t - u_{xx} + u u_x = 0 \)
   
   (d) \( u_{tt} + u_{xxxx} + \sqrt{1 + u} = 0 \)

5. Classify the following second order linear PDEs. Note that for for variable coefficients, you will need to characterize by region in the \((x,y)\) plane.
   
   (a) \( xu_{xx} - 4u_{xy} = 0 \)
   
   (e) \( \sin(xy)u_{xy} = 0 \)

   (b) \( x^2 u_{xx} - y^2 u_{yy} = 0 \)
   
   (f) \( 2yu_{xx} + (x + y)u_{xy} + 2xu_{yy} = 0 \)

6. (DuZ, 1.3.7,8) Suppose that \( D \) is the unit square \((0,1) \times (0,1)\). Verify that:
   
   (a) both \( u_1(x,y) = \sin(\pi x) \sinh(\pi y) \) and \( u_2(x,y) = \sin(2\pi x) \sinh(2\pi y) \) satisfy the conditions

   \[
   \begin{align*}
   \Delta u &= 0 & \text{in } D \\
   u(x = 0) &= 0, & u(x = 1) &= 0 & 0 < y < 1 \\
   u(y = 0) &= 0 & 0 < x < 1.
   \end{align*}
   \]

   (b) Is this previous part a well-posed boundary value problem? Explain your answer.
7. (DuZ, 1.4.7,8,+) Consider the wave equation

\[ u_{tt}(x, t) = c^2 u_{xx}(x, t). \]

(a) For any positive integer \( m \), show that each of the following functions satisfies the wave equation for all \( x, t \):

\[
\begin{align*}
u_1(x, t) &= \sin(m\pi x) \sin(m\pi ct) \\
u_2(x, t) &= \sin \left((m + 1/2)\pi x\right) \cos \left((m + 1/2)\pi ct\right)
\end{align*}
\]

(b) Now let \( g : \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function. Show that

\[ u(x, t) = \int_{x-ct}^{x+ct} g(s) \, ds \]

is a solution to the same wave equation above.

(c) Each of \( u_i \) for \( i = 1, 2 \) in part (a) satisfies at least one of the following initial or boundary condition. Match each function with all of the conditions that it satisfies.

i. \( u(0, t) = 0 \)

ii. \( u_x(0, t) = 0 \)

iii. \( u(x, 0) = 0 \)

iv. \( u_t(x, 0) = 0 \)

8. (a) Find a general solution to the PDE \( u_{xy} = 0 \).

(Give your answer in terms of two arbitrary functions.)

(b) Solve the nonlinear PDE in the function \( u : \mathbb{R}^2 \to \mathbb{R} \):

\[ (\Delta u)^2 - 2u_{xx}u_{yy} = 0. \]

Give your answer as general as possible, and note that this solution does not contain arbitrary functions (though it will have arbitrary coefficients). What is the key difference between this PDE and that in part (a)\

9. Linear, homogeneous PDEs with constant coefficients admit complex-valued solutions (called \textit{wave trains}) of the form

\[ u(x, t) = Ae^{i(kx-\omega t)}, \]

where \( i = \sqrt{-1} \) is the imaginary unit. The constant \( A \) is the amplitude, \( k \) is the wave number, and \( \omega \) is the temporal frequency. We often substitute the plane wave form (an Ansatz) into a PDE, a \textit{dispersion relation} of the form \( \omega = \omega(k) \) results. This relation tells us how frequency depends on wave number.

For each of the following PDEs, find the associated dispersion relation.

(a) \( u_t = Du_{xx} \)

(b) \( u_{tt} = c^2 u_{xx} \)

(c) \( u_t = -u_{xxx} \)

(d) \( u_t = ihu_{xx} \)

(e) \( u_t + cu_x = 0 \)
10. (Bonus) We can (yet again) classify PDEs by their dispersion relations. If $\omega(k)$ is complex (with both real and imaginary parts), the PDE is called diffusive. If $\omega(k)$ is real and $u''(k) \neq 0$, the PDE is called dispersive. The heat (diffusion) equation is diffusive (check this), and the wave equation is neither diffusive nor dispersive. The term dispersive means that the plane wave speed $\omega(k)/k$ changes with $k$, i.e., waves of different wavelength travel at different speeds, and thus they disperse.

Classify each of the PDEs in the previous problem according to this scheme.

11. (Bonus) Solve the ODE (general form) from 1a.

12. (Bonus) Show that $e^{-\xi y} \sin(\xi x)$, $x \in \mathbb{R}$, $y > 0$, is a solution to Laplace’s equation $u_{xx} + u_{yy} = 0$ for all values of $\xi \in \mathbb{R}$. Deduce that

$$\int_0^\infty c(\xi)e^{-\xi y} \sin(\xi x)d\xi$$

is also a solution to Laplace’s equation, so long as $c(\xi)$ is a bounded, continuous function on $[0, \infty)$. These hypotheses are sufficient to differentiate through the integral.

Explain, in a couple of sentences, how this result is related to the principle of superposition. (Think about the fact that integration is arrived at via a limit of sums.)

13. (Bonus) (DuZ, 1.6.2) Consider the equation

$$F(\mu) = \mu^2 - (A + C)\mu - (B^2 - AC) = 0.$$  

Note that the graph of $F(\mu) = 0$ is a parabola (in $\mu$) opening upward. Demonstrate the following:

(a) If $B^2 - AC < 0$, then $F(\mu) = 0$ has two real roots of the same sign. (Consider cases.)

(b) If $B^2 - AC = 0$, then $F(\mu) = 0$ has $\mu = 0$ as a root. (What determines the sign of the other root?)

(c) If $B^2 - AC > 0$, then $F(\mu) = 0$ has distinct roots of opposite sign.