Due: W, 9/11 in class

This assignment is meant to illustrate the mathematical ideas, notation, and tools that will be used in this class. It isn’t intended to be particularly difficult—though it will be time consuming. It should help you refresh on concepts that may have become a bit fuzzy. *It is likely necessary, and encouraged, that you look up definitions or formulae to assist you.*

Everyone will write up individual solutions, *showing all details.* Where explanations/discussions are requested, complete sentences are expected. Scratchwork will not be graded. Typesetting solutions would be best (see the syllabus). One should complete most of most of the questions for full points.

1. Integration.
   (a) Assuming \(x > 0\), compute, showing all work (not making use of a table of integrals)
   \[
   \int_0^\infty (t-3)e^{-tx} \, dt
   \]
   (b) Suppose that \(f, g\) are smooth functions that have the property that
   \(f(0) = f'(0) = g(0) = g'(0) = 0\). Show that
   \[
   \int_0^L f^{(4)}(x)g(x) \, dx = \int_0^L f''(x)g''(x) \, dx + f'''(L)g'(L) - f''(L)g'(L).
   \]
   (c) Showing your work, compute the following integrals for all integers \(1 \leq m, n \leq 3\):
   \[
   \frac{1}{\pi} \int_{-\pi}^\pi \sin(mx) \sin(nx) \, dx
   \]
   Note, there are 9 total values you should record (though, by symmetry, some are repeated). These are not particularly challenging integrals, and a pattern will become clear (indicated this). Present your answer as a \(3 \times 3\) matrix/array.

   **Make a general conjecture** (in the form of a couple of sentences) about integral relations for the family of functions \(\left\{ \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^\infty\).

2. Series.
   (a) For each of the following series, determine: if it converges **C**, or if it diverges **D**. Then state in a sentence or two how you arrived at your conclusion.
   \[
   \begin{align*}
   \text{i. } & \sum_{n=1}^\infty \pi^{-n} & \text{ii. } & \sum_{n=2}^\infty \frac{n}{n^2 - 1} & \text{iii. } & \sum_{n=1}^\infty \frac{\cos(n)}{n^2}
   \end{align*}
   \]
   (b) Provide the corresponding function represented by this power series on its interval of convergence, then state the interval of convergence.
   \[
   \sum_{n=0}^\infty \frac{1}{(n+1)}(2x)^{n+1}
   \]

(a) Suppose that $\phi$ is a smooth scalar function and $\vec{F}$ is a smooth vector field. Which of the following operations makes sense? (Just indicate “defined” or “undefined” for each bullet, no explanation necessary. Note that here we interpret $\nabla$ to be the standard gradient operator, i.e., $\langle \partial_j \rangle^T$—not to be confused with the Jacobian matrix.)

i. $\nabla \cdot \phi$

ii. $\nabla (\nabla \phi)$

iii. curl ($\nabla \phi$)

iv. $\nabla \vec{F}$

v. $\nabla (\text{div} \vec{F})$

vi. $\nabla (\nabla \times \phi)$

vii. $\nabla \times (\text{div} \vec{F})$

viii. $\nabla \cdot \phi$

ix. $\nabla \cdot (\nabla \cdot \vec{F})$

x. curl (curl $\vec{F}$)

(b) Write out the product rule for the divergence of a scalar-vector product:

$$\text{div} (u\vec{F}) = ...$$

where $u$ is a scalar function. (You might try to check it component-wise in 2-D.)

(c) Kernels of some operations.

i. Let $\vec{F}$ be a (smooth) conservative vector field with potential function $\phi$. What must be true of $\text{curl} \vec{F} = \nabla \times \vec{F} = \nabla \times \nabla \phi$? In other words, the curl of a conservative vector field is...(finish the sentence and justify.)

ii. Let $\vec{C} = \nabla \times \vec{F}$ ($\vec{C}$ is a curl field). What must be true of $\text{div} \vec{C} = \nabla \cdot \vec{C} = \nabla \cdot (\nabla \times \vec{F})$? In other words, the divergence of a curl field must be...(finish and justify.)

4. Integration, II.

Recall that for any smooth, oriented curve $\Gamma \subset \mathbb{R}^2$ we can define the unit radial (with positive orientation) vector $\vec{r}(x, y)$ and the unit outward normal vector $\vec{n}(x, y)$. For all values along $\Gamma$, we have $d\vec{r} \cdot d\vec{n} = 0$, where $d\vec{r} = \langle dx, dy \rangle$ and $d\vec{n} = \langle dy, -dx \rangle$.

(a) Consider $C_1$ to be the semicircle of radius two in the upper half plane with standard orientation (counter clockwise). Compute the following in any way you’d like.

i. The average value of the function $f(x, y) = x + y + 2$ on $C_1$.

ii. The flux through $C_1$ of the vector field $\vec{F}(x, y) = \langle x^2y, (1/3)x^3 + y \rangle$

iii. Now consider $C_2$ to be the line segment going from $(-2, 0)$ to $(2, 0)$. Compute the circulation of $\vec{F}$ (as given above) over the closed curve $C = C_1 \cup C_2$.

5. Areas.

(a) Compute the surface area (showing all work) of a spherical cap with height $h$ cut from a sphere with radius $R$.

(b) Use a line integral to compute the area bounded by an ellipse $C$ parametrized by

$$\vec{r}(t) = \langle 2\cos(t), 7\sin(t) \rangle, \quad t \in [0, 2\pi).$$

(You will want to use the divergence theorem in 2-D and choose a smart vector field $\vec{F}$.)
6. The Laplacian.

(a) Recall that for \( f(x, y) \) we have \( \Delta f = \nabla \cdot \nabla f \). In Cartesian coordinates, \( \Delta f = f_{xx} + f_{yy} \).

Consider polar coordinates with the change of variable mapping:

\[
\begin{align*}
\theta(x, y) &= \arctan(y/x) \\
r(x, y) &= \left[x^2 + y^2\right]^{1/2}.
\end{align*}
\]

Thinking of \( f \) as \( f(r, \theta) \), and using the chain rule, compute the expression for \( \Delta f \) in terms of \( r \) and \( \theta \)—show all work.

(b) Let \( \Omega \) be a region bounded by a curve \( \Gamma \), where \( \Gamma \) is a positively oriented, p.w. smooth, simple, closed curve in \( \mathbb{R}^2 \). (These are the hypotheses for Green’s Theorem.) Recall that function \( u \) is called harmonic if \( \Delta u = 0 \).

i. Argue that, if \( u \) is harmonic, then \( \oint_\Gamma \nabla u \cdot \mathbf{n} \, ds = 0 \) (line integral over \( \Gamma \) is zero).

ii. Argue that, if \( u \) is harmonic and \( u(x, y) = 0 \) on \( \Gamma \), then \( \iint_\Omega \nabla u \cdot \nabla u \, dA = 0 \). What can you infer about \( u \) in this case?

7. First order ODE.

(a) Verify that \( y(x) = x[1 + \cos(x)] \) solves the IVP:

\[
\frac{dy}{dx} = \frac{y}{x} - x \sin(x), \quad y(\pi) = 0.
\]

(b) Solve the ODE:

\[t^3 y' + 4t^2 y = e^{-t}, \quad t > 0.\]

8. Modeling.

Consider a baseball of mass \( 0.2 \, \text{kg} \) thrown directly towards the ground with initial velocity \( 30 \, \text{m/s} \) from the top of a very tall building. Assume the force of air resistance is proportional (with proportionality constant \( 1 \, \text{kg} \cdot \text{s}^{-1} \)) to the instantaneous velocity.

Write and solve an initial value problem describing its velocity. What happens as \( t \to \infty \)?

9. Second order constant coefficient ODE.

(a) Consider the second order, constant coefficient differential equation in \( x(t) \):

\[
x'' + bx' + 2x = 0. \tag{1}
\]

Find a number \( b \) so that solutions are damped, but not overdamped—i.e., such that solutions \( x(t) \to 0 \) as \( t \to \infty \) but still have a finite period associated to them.

(b) Consider the second order, constant coefficient differential equation in \( x(t) \):

\[
x'' - 4x' + 4x = g(t). \tag{2}
\]

i. Let \( g(t) = 2e^t \). What is the general solution to (2) in this (inhomogeneous) situation?

ii. Let \( g(t) = 0 \) and \( x(0) = 1, \ x'(0) = 0 \).

In this (homogeneous) case what is the solution satisfying initial value problem?

iii. Let \( g(t) = \sin(t) + e^{2t} \).

What is the appropriate form for the particular solution to (2) in this case?

(Do not solve for undetermined coefficients.)

(c) Consider the forced ODE

\[
x'' + 9x = \sin(\omega t). \]

Find a value of \( \omega \) such that resonance occurs, and explain what this means.