1. Integration.

(a) Compute, showing all work (not making use of a table of integrals)
\[ \int_0^\infty (t - 3) e^{-tx} \, dt \]
First consider the integral
\[ \int_0^M (t - 3) e^{-tx} \, dt. \]
Via integration by parts \( u = (t - 3), \ dv = e^{-tx} \), we have
\[ \int_0^M (t - 3) e^{-tx} \, dt = \frac{t - 3}{x} e^{-tx}\bigg|_{t=0}^{t=M} - \int_0^M \frac{1}{x} e^{-tx} \, dt = \frac{t - 3}{x} e^{-tx}\bigg|_{t=0}^{t=M} - \frac{1}{x^2} e^{-tx}\bigg|_{t=0}^{t=M}. \]
Thus
\[ \int_0^M (t - 3) e^{-tx} \, dt = \frac{M - 3}{x} e^{-Mx} + 3 - \frac{1}{x^2} e^{-Mx} + \frac{1}{x^2}. \]
Taking the limit as \( M \to \infty \) and using l'Hospital we arrive at the improper Riemann integral, and have
\[ \int_0^\infty (t - 3) e^{-tx} \, dt = 3 + \frac{1}{x^2}, \]
requiring that \( x > 0 \) (otherwise the integral doesn’t converge).

(b) Suppose that \( f, g \) are smooth functions that have the property that \( f(0) = f'(0) = g(0) = g'(0) = 0 \). Show that
\[ \int_0^L f^{(4)}(x)g(x) \, dx = \int_0^L f''(x)g''(x) \, dx + f'''(L)g(L) - f''(L)g'(L). \]
We will use the version of integration by parts that reads as:
\[ \int_a^b f(x)g'(x) \, dx = f(x)g(x)\bigg|_{x=a}^{x=b} - \int_a^b f'(x)g(x) \, dx. \]
We have
\[ \int_0^L f^{(4)}(x)g(x) \, dx = f''(x)g''(x)\bigg|_{x=0}^{x=L} - \int_0^L f'''(x)g'(x) \, dx \]
invoking the BC \( g(0) = 0 \)
\[ = f''(L)g(L) - \int_0^L f'''(x)g'(x) \, dx \]
integrating by parts again and invoking the BC \( g'(0) = 0 \)
\[ = f''(L)g(L) - f''(L)g'(L) + \int_0^L f''(x)g''(x) \, dx \]
(c) Showing your work, compute each of the following integrals for all integers \( 1 \leq m, n \leq 3 \):

i. \( \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx \)

ii. \( \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx \)

Let’s adopt a convenient notation:

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx \equiv (f,g).
\]

For \((\sin(mx), \cos(nx))\) the associated 9 integrals can be computed using basic trig integrations (with trig identities like double and half angle formulas). Instead, let us note the product to sum formulae:

\[
\sin(mx) \cos(nx) = \frac{1}{2} \left[ \sin((m + n)x) - \sin((m - n)x) \right]
\]

\[
\sin(mx) \sin(nx) = \frac{1}{2} \left[ \cos((m - n)x) - \cos((m + n)x) \right]
\]

When \(m = n\), these collapse into the more familiar

\[
\sin(mx) \cos(mx) = \frac{1}{2} \sin(2mx)
\]

\[
\sin^2(mx) = \frac{1}{2} \left[ 1 - \cos(2mx) \right]
\]

Each of the above can be integrated using elementary techniques (the trig terms integrate away on \([-\pi, \pi]\)). Then, using our notation, we have (letting \(m\) dictate the row and \(n\) the column)

\[
\begin{bmatrix}
(\sin(mx), \cos(nx))_{m,n=1,2,3}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
(\sin(mx), \cos(nx))_{m,n=1,2,3}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The families \(\{\sin(nx)\}_{n=1}^{\infty}\) and \(\{\cos(nx)\}_{n=1}^{\infty}\) are orthogonal to one another. If we choose any function from either family and integrate it against any other in terms of \((f,g)\) we get zero. If we integrate two distinct functions from the same family, those are also zero. The only time we get something other than zero (1, in fact) is when we form \((\sin(mx), \sin(mx))\) or \((\cos(nx), \cos(nx))\).
2. Series.

(a) For each of the following series, determine: if it absolutely converges AC, if it converges but not absolutely C, or if it diverges D. Then state in a sentence or two how you arrived at your conclusion.

i. \( \sum_{n=1}^{\infty} \pi^{-n} \)

AC. This is a geometric series that converges since \( \frac{1}{\pi} < 1 \). In particular, it converges to \( \frac{1}{\pi - 1} \). Since it is a positive series that converges, it converges absolutely.

ii. \( \sum_{n=2}^{\infty} \frac{n}{n^2 - 1} \)

D. This series diverges, for instance, using the limit comparison test or the direct comparison test with the harmonic series.

iii. \( \sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \)

AC. Note that \( \left| \frac{\cos(n)}{n^2} \right| \leq \frac{1}{n^2} \). We have that \( \sum \frac{1}{n^2} \) converges by the p-test (or the integral test), and thus \( \sum \left| \frac{\cos(n)}{n^2} \right| \) converges by comparison.

D. We consider the ratio test and form \( \frac{a_{n+1}}{a_n} \).

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^{n+1}} = \frac{(n+1)(n+1)^n}{(n+1)n!} \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n.
\]

Thus \( \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim(1 + 1/n)^n = e > 1 \), and so the series diverges by the ratio test.

(b) For each of the following power series, provide the corresponding function represented by that power series on its interval of convergence.

i. \( \sum_{n=0}^{\infty} \frac{1}{(n+1)} x^{n+1} \)

Recall the series relation \( \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad x \in (-1,1) \). We can integrate a power series identity on the interior of the interval of convergence. Let us integrate both sides to obtain

\[
\sum_{n=0}^{\infty} \int x^n \, dx = \int \frac{1}{1-x} \, dx,
\]

yielding

\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C.
\]
By choosing \( x = 0 \), we deduce that the constant of integration is \( C = 0 \). Reindexing the sum, we have
\[
\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln |1-x|, \quad x \in (-1, 1).
\]
We have to check the endpoints of the interval. In doing so, we note that \( x = -1 \) yields a convergent
alternating series, and thus the interval of convergence is actually \([-1, 1)\), giving
us the interesting relation \( \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = \ln(.5) \).


(a) Suppose that \( \phi \) is a smooth scalar function and \( \vec{F} \) is a smooth vector field. Which
of the following operations makes sense? (Simply indicate on your paper by writing
“defined” or “undefined” next to the appropriate number.)

i. \( \nabla \cdot \phi \)
   Defined. \( \phi \) is scalar and the divergence eats vectors and produces scalars.

ii. \( \nabla(\nabla \phi) \)
   Undefined, ambiguous. In this class we will think of the gradient as being an
operation that eats a scalar function of \( n \) independent variables and produces an
\( n \)-vector. (There is a more general notion, typically written as \( D \), which general-
izes the gradient of a vector function by taking the gradient of each component
of a vector function. This is not what we mean here.)

iii. curl \( (\nabla \phi) \)
   Defined. The curl, \( \nabla \times \) is an operation that eats 3-vectors and spits out 3-
vectors. It measures the tendency of a vector field to rotate, in some sense. Note
that the curl of a potential field is zero (see later exercise).

iv. \( \nabla \vec{F} \)
   Undefined, ambiguous. As before, the gradient eats scalars in this class.

v. \( \nabla(\text{div} \vec{F}) \)
   Defined. The divergence takes the field \( \vec{F} \) and makes it a scalar, at which
point the gradient can be applied. This collection of operations is in some sense complementary (or “dual”) to the operation \( \Delta = \text{div} \nabla \), which acts on scalars.

vi. \( \nabla(\nabla \times \phi) \)
   Undefined. The curl operates on vector fields, and thus, since \( \nabla \cdot \vec{F} \) is a scalar,
the operation is undefined.

vii. \( \nabla \times (\text{div} \vec{F}) \)
   Undefined. The curl operates on vector fields, not scalars.

viii. \( \nabla \cdot \nabla \phi \)
   Defined. As we’ve seen, this operation is the Laplacian, which can be written
in a variety of ways. It acts on scalars.

ix. \( \nabla \cdot (\nabla \cdot \vec{F}) \)
   Undefined. The divergence doesn’t know what to do with a scalar, and since
\( \nabla \cdot \vec{F} \) (the divergence) is a scalar, it is confused.

x. curl (curl \( \vec{F} \))
**Defined.** Since the curl operates on vector fields, and the curl of a vector field is a vector field, this is a fine operation. There is even a cool vector identity:

\[ \nabla \times (\nabla \times \vec{F}) - \nabla (\nabla \cdot \vec{F}) = - (\Delta f_1, \Delta f_2, \Delta f_3), \]

where the \( f_i \) are the components of \( \vec{F} \).

(b) Prove the following rule for the divergence:

\[ \text{div} (u\vec{F}) = u(\text{div} \vec{F}) + \nabla u \cdot \vec{F}, \]

where \( u \) is a scalar function. (Checking it in 2-D is alright.) Let’s use the convention that the vector \( \vec{v} \) has components \( v_i \). Note, then that \( u\vec{F} \) is a vector with components \( (u\vec{F})_i = uf_i \). Thus, when computing a partial we will be looking at \( \partial_j(uf_i) \). Noting that both of these quantities are scalars, we invoke the scalar product rule (from Calc. 1).

\[ \partial_j(uf_i) = (\partial_j u)f_i + u(\partial_j f_i). \]

Thus

\[ \text{div} (u\vec{F}) = \sum_j \partial_j(uf_j) = \sum_j \partial_j uf_j + \sum_j u\partial_j f_j = \nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F}), \]

by the definition of the divergence.

(c) Kernels of some operations.

i. Let \( \vec{F} \) be a (smooth) conservative vector field with potential function \( \phi \). What must be true of \( \text{curl} \vec{F} = \nabla \times \vec{F} = \nabla \times \nabla \phi? \)

In other words, the curl of a conservative vector field is...(finish this sentence and justify your claim.)

The curl of a conservative vector field is the zero vector! We can check this by direct computation, using the equality of mixed partials. Indeed, since \( \vec{F} \) is conservative, \( \vec{F} = \nabla \phi \) for some scalar function \( \phi \). So \( \vec{F} = (\phi_x, \phi_y, \phi_z) \). Recalling that the curl of a 3-D vector field is

\[ \text{curl} \vec{F} = \nabla \times \vec{F} = (\partial_y f_3 - \partial_z f_2)\hat{i} - (\partial_z f_3 - \partial_x f_1)\hat{j} + (\partial_x f_2 - \partial_y f_1)\hat{k}. \]

Now, in this case, this reduces to

\[ \text{curl} \vec{F} = (\partial_y \phi_z - \partial_z \phi_y)\hat{i} - (\partial_z \phi_z - \partial_x \phi_z)\hat{j} + (\partial_x \phi_y - \partial_y \phi_x)\hat{k} = \vec{0}. \]

ii. Let \( \vec{C} = \nabla \times \vec{F} \). What must be true of \( \text{div} \vec{C} = \nabla \cdot \vec{C} = \nabla \cdot (\nabla \times \vec{F})? \)

In other words, the divergence of a curl field must be...(finish this sentence and justify your claim.)

The divergence of a curl field must be zero! We again check by direct computation using the above description of the curl and, again, the equality of mixed partials.

\[ \nabla \cdot \text{curl} \vec{F} = \partial_x (\partial_y f_3 - \partial_z f_2) - \partial_y (\partial_x f_3 - \partial_z f_1) + \partial_z (\partial_x f_2 - \partial_y f_1) = 0 \]

after inspection.
4. Integration, II.

Recall that for any smooth, oriented curve $\Gamma \subset \mathbb{R}^2$ we can define the unit radial (with positive orientation) vector $\vec{r}(x,y)$ and the unit outward normal vector $\vec{n}(x,y)$. For all values along $\Gamma$, we have $d\vec{r} \cdot d\vec{n} = 0$, where $d\vec{r} = \langle dx, dy \rangle$ and $d\vec{n} = \langle dy, -dx \rangle$.

(a) Consider $C_1$ to be the semicircle of radius two in the upper half plane with standard orientation (counter clockwise). Compute the following in any way you’d like.

i. The average value of the function $f(x,y) = x + y + 2$ on $C_1$.

We define the average value to be $\overline{f} = \frac{\int_{C_1} f(x,y)ds}{|C_1|}$.

Since $C_1$ is the semicircle, we know the length of $C_1 = 2\pi$. To compute the line integral, we utilize a standard parameterization with $x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$, $t \in [0,\pi]$. Recalling the arc length element $ds = (dx^2 + dy^2)^{1/2}$, we can invoke the parametrization, as well as $x'(t) = -2\sin(t)$, $y'(t) = 2\cos(t)$.

$$\int_{C_1} (x + y + 2)ds = \int_0^\pi [2\cos(t) + 2\sin(t) + 2]\sqrt{4\cos^2(t) + 4\sin^2(t)}dt.$$ 

Invoking the pythagorean trig identity and integrating, we arrive at $8 + 4\pi$. Thus $\overline{f} = \frac{4 + 2\pi}{\pi}$.

ii. The flux through $C_1$ of the vector field $\vec{F}(x,y) = \langle x^2y, (1/3)x^3 + y \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{n} = \int_{C_1} \langle x^2y, (1/3)x^3 + y \rangle \cdot \langle dy, -dx \rangle = \int_{C_1} x^2ydy - (\frac{1}{3}x^3 + y) dx$$

We now invoke the parametrization, as above: Using elementary integration, we have that the integral equals $-2\pi$.

$$\int_{C_1} x^2ydy - (\frac{1}{3}x^3 + y) dx = \int_0^\pi [(32/3)\cos^3(t)\sin(t) - 4\sin^2(t)]dt$$

iii. Now consider $C_2$ to be the line segment going from $(-2,0)$ to $(2,0)$. Compute the circulation of $\vec{F}$ (as given above) over the closed curve $C = C_1 \cup C_2$.

This is a closed curve, and one can quickly check that $\vec{F}$ is a conservative vector field. The circulation of a conservative vector field is always zero, by the fundamental theorem of line integrals.

5. Areas.

(a) Compute the surface area (showing all work) of a spherical cap with height $h$ cut from a sphere with radius $\rho$.

In general, we are interested in the surface integral:

$$\iint_S (1)dS = \iint_T 1||\vec{r}_u \times \vec{r}_v||dudv,$$
where \( S \) is the surface in question, and \( \vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \) is a parametrization of \( S \) over the region \( (u, v) \in T \). (The symbol \( \times \) is the cross product; \( ||\vec{v}|| \) refers to the norm of the vector \( \vec{v} \) in \( \mathbb{R}^3 \).)

To compute a surface integral, we need to parameterize \( \vec{r} = \langle x, y, z \rangle \) the upper hemisphere. This is a standard parametrization (coming from the Spherical to Cartesian change of variables—\( u \) is traditional angular variable in the \( x-y \) plane, and \( v \) is the polar angle, emanating from \( \vec{k} \to -\vec{k} \)):

\[
\begin{align*}
    x(u, v) &= \sqrt{\rho} \sin(v) \cos(u) \\
    y(u, v) &= \sqrt{\rho} \sin(v) \sin(u) \\
    z(u, v) &= \sqrt{\rho} \cos(v)
\end{align*}
\]

Thus (after some extensive calculation with calculus and trig identities):

\[
\begin{align*}
    \vec{r}_u &= \langle -\rho \sin(v) \sin(u), \rho \sin(v) \cos(u), 0 \rangle \\
    \vec{r}_v &= \langle \rho \cos(v) \cos(u), \rho \cos(v) \sin(u), -\rho \sin(v) \rangle \\
    \vec{r}_u \times \vec{r}_v &= \langle -\rho^2 \sin(v) \cos(u), -\rho^2 \sin^2(v) \sin(u) - \rho^2 \sin(v) \cos(v), \rho^2 \sin(v) \rangle \\
    ||\vec{r}_u \times \vec{r}_v|| &= \rho^2 \sin(v).
\end{align*}
\]

Now we need to determine what values of \( u, v \) between 0 and \( \pi \) parameterize the cap. If we allow all values of \( 0 \leq v \leq \pi \) we will get the hemisphere. Doing some trig reveals that for radius \( \rho \) and height \( h \), the parameter range we are interested in is \( 0 \leq v \leq \arccos \left(1 - \frac{h}{\rho}\right)\). Thus the double integral we want is:

\[
\int_0^{2\pi} \int_0^{\arccos(1-h/\rho)} \rho^2 \sin(v) dv du = -2\pi \rho^2 \left[ \cos(v) \right]_0^{\arccos(1-h/\rho)} = 2\pi \rho h.
\]

(b) Use a line integral to compute the area bounded by an ellipse \( C \) parametrized by

\[
\vec{r}(t) = \langle 2 \cos(t), 7 \sin(t) \rangle, \quad t \in [0, 2\pi].
\]

Recall

\[
\oint_C \vec{F} \cdot d\vec{n} = \iint_D (\text{div} \vec{F}) dA.
\]

Using the RHS of the integral above, we choose a vector field \( \vec{F}(x, y) = \frac{1}{2} \langle x, y \rangle \), whose divergence is the scalar value 1. Thus for this choice of vector field, the above integral identity reads:

\[
\frac{1}{2} \int_C \langle x, y \rangle \cdot (dy, -dx) = \iint_D 1 dA,
\]

where we have recalled that \( d\vec{n} = \langle dy, -dx \rangle \) in this case (and thus \( d\vec{n} \cdot d\vec{r} = 0 \), with \( d\vec{r} = \langle dx, dy \rangle \)). The line integral simplifies via the dot product and the integrating the function 1 over a region returns area, thus we have the identity:

\[
\frac{1}{2} \int_C xdy - ydx = A_{\text{region}}.
\]
Invoking the given parametrization for $x$ and $y$, we note that $dx = -2\sin(t)$ and $dy = 7\cos(t)$, yielding

$$A = \frac{1}{2} \int xdy - ydx = \frac{1}{2} \int_0^{2\pi} 2\cos(t)(7\cos(t)) - 7\sin(t)(-2\sin(t))dt.$$ 

Using the Pythagorean identity $\sin^2(t) + \cos^2(t) = 1$, we have

$$A = 14\pi.$$  

6. The Laplacian.  

(a) Recall that for $f(x, y)$ we have $\Delta f = \nabla \cdot \nabla f = f_{xx} + f_{yy}$. Consider polar coordinates with the change of variable mapping:

$$\theta(x, y) = \arctan(y/x)$$

$$r(x, y) = \left[x^2 + y^2\right]^{1/2}.$$ 

Thinking of $f$ as $f(r, \theta)$, and using the chain rule, compute the expression for $\Delta f$ in terms of $r$ and $\theta$. 

Here we will also need to note the inverse relations:

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$ 

We first note that by the chain rule

$$\partial_x f = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x},$$

which can be simplified by computing the partials of $r, \theta$. After some work, we see that

$$r_x = \frac{x}{(x^2 + y^2)^{1/2}}, \quad \theta_x = \frac{-y}{x^2 + y^2}.$$ 

Invoking the inverse relations above, we have

$$r_x = \cos(\theta), \quad \theta_x = \frac{-\sin(\theta)}{r}.$$ 

Thus the differentiation in $x$ goes as

$$\frac{\partial f}{\partial x} = f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r}.$$ 

A similar calculation shows that

$$\frac{\partial f}{\partial y} = f_r \sin(\theta) + f_\theta \frac{\cos(\theta)}{r}.$$ 

We can then iterate to find the necessary second partials:

$$\partial^2_x f = \partial_x(f_x) = \partial_x \left[f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r}\right].$$
\[ \frac{\partial^2 f}{\partial x^2} = \partial_x \left[ f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r} \right] \]
\[ = \partial_r \left[ f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r} \right] \cos(\theta) - \partial_\theta \left[ f_r \cos(\theta) - f_\theta \frac{\sin(\theta)}{r} \right] \sin(\theta) \]
\[ = f_{rr} \cos^2(\theta) - f_{r\theta} \frac{\sin(\theta) \cos(\theta)}{r} + f_{\theta \theta} \frac{\sin^2(\theta)}{r^2} \]
\[ + f_r \frac{\sin^2(\theta)}{r} - f_{r\theta} \frac{\cos(\theta) \sin(\theta)}{r} + f_{\theta \theta} \frac{\cos(\theta) \sin(\theta)}{r^2} \]

Similarly, in \( y \), we have:
\[ \frac{\partial^2 f}{\partial y^2} = \partial_y \left[ f_r \sin(\theta) + f_\theta \frac{\cos(\theta)}{r} \right] \]
\[ = \partial_r \left[ f_r \sin(\theta) + f_\theta \frac{\cos(\theta)}{r} \right] \sin(\theta) + \partial_\theta \left[ f_r \sin(\theta) + f_\theta \frac{\cos(\theta)}{r} \right] \cos(\theta) \]
\[ = f_{rr} \sin^2(\theta) + f_{r\theta} \frac{\sin(\theta) \cos(\theta)}{r} - f_{\theta \theta} \frac{\sin(\theta) \cos(\theta)}{r^2} \]
\[ + f_r \frac{\cos^2(\theta)}{r} + f_{r\theta} \frac{\cos(\theta) \sin(\theta)}{r} + f_{\theta \theta} \frac{\cos(\theta) \sin(\theta)}{r^2} \]

Putting it all together, and using the Pythagorean identity:
\[ \Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \theta^2} = \frac{1}{r} \partial_r \left[ r \partial_r f \right] + \frac{1}{r^2} \partial^2_{\theta \theta} f. \]

(b) Let \( \Omega \) be a region bounded by a curve \( \Gamma \), where \( \Gamma \) is a positively oriented, p.w. smooth, simple, closed curve in \( \mathbb{R}^2 \). (Note these are the hypotheses for Green’s Theorem.) Recall that function \( u \) is called harmonic if \( \Delta u = 0 \) (in any dimension).

i. Argue that, if \( u \) is harmonic, then
\[ \oint_{\Gamma} \nabla u \cdot \vec{n} \, ds = 0. \]

This is a direct application of the divergence theorem, using the description of the Laplacian as \( \Delta = \text{div} \, \nabla \). Since we have assumed that \( \Delta u \equiv 0 \) on the region \( \Omega \), we can integrate this scalar function to arrive at
\[ \iint_{\Omega} (\Delta u) dA = 0. \]

On the other hand, invoking the divergence theorem, we have:
\[ \iint_{\Omega} (\Delta u) dA = \iint_{\Omega} \text{div} \, \nabla u \, dA = \oint_{\Gamma} \nabla u \cdot \vec{n} ds. \]

Putting the two identities together, we have the desired result.

ii. Argue that, if \( u \) is harmonic and \( u(x, y) = 0 \) on \( \Gamma \), then
\[ \iint_{\Omega} \nabla u \cdot \nabla u \, dA = 0. \]
What can you infer about $u$ in this case?
Let’s begin by recalling the relation (using the product rule from an earlier problem):

$$\text{div} \ (u \nabla u) = \nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u(\nabla \cdot \nabla u) = |\nabla u|^2 + u(\Delta u).$$

Thus, integrating we have:

$$\iiint_{\Omega} \text{div} \ (u \nabla u) \, dA = \iiint_{\Omega} \nabla u \cdot \nabla u \, dA + \iiint_{\Omega} u(\Delta u) \, dA.$$

We invoke the divergence theorem on the first term, as well as the hypothesis that $u \equiv 0$ on $\Gamma$:

$$\iiint_{\Omega} \text{div} \ (u \nabla u) \, dA = \oint_{\Gamma} u(\nabla u \cdot \vec{n}) \, ds = \oint_{\Gamma} (0)(\nabla u \cdot \vec{n}) \, ds = 0.$$

On the other hand, since we assumed that $u$ was harmonic, we know that $\Delta u \equiv 0$ in $\Omega$, and thus

$$\iiint_{\Omega} u(\Delta u) \, dA = \iint_{\Omega} u(0) \, dA = 0.$$

Putting these together, we have that

$$\iiint_{\Omega} \nabla u \cdot \nabla u \, dA = 0.$$

(Since $\nabla u \cdot \nabla u = |\nabla u|^2 \geq 0$, we know that $|\nabla u| = 0$, and this can only happen if $u = \text{constant}$. But since we know that $u = 0$ on $\Gamma$, we can conclude that $u \equiv 0$ on the entirety of $\Omega$.)

7. First order ODE.

(a) Verify that $y(x) = x[1 + \cos(x)]$ solves the IVP:

$$\frac{dy}{dx} = \frac{y}{x} - x \sin(x), \quad y(\pi) = 0.$$

For our purported solution, we have $\frac{dy}{dx} = [1 + \cos(x)] - x \sin(x)$. In addition, we have that $\frac{y}{x} = [1 + \cos(x)]$. So $\frac{dy}{dx} - \frac{y}{x} = -x \sin(x)$, which confirms that this $y$ satisfies the ODE. We must also check the initial condition: $y(\pi) = \pi[1 + \cos(\pi)] = \pi[1 - 1] = 0$. So the initial condition is satisfied as well.

(b) Solve the ODE

$$t^3 y' + 4t^2 y = e^{-t}, \quad t > 0.$$

Since $t > 0$, divide through by $t^3$ to obtain

$$y' + \frac{4}{t} y = t^{-3} e^{-t}.$$
In this case (first order, linear ODE) we can invoke the integrating factor. Here, the integrating factor is $\mu = \exp\left(\int 4t^{-1} \, dt\right) = e^{4\ln|t|} = t^4$. The ODE can then be written as: $(t^4y)' = te^{-t}$. We can then integrate (by parts) to arrive at
\[ t^4y = -te^{-t} + e^{-t} + C, \]
which can be rewritten as
\[ y(t) = t^{-4}e^{-t} - t^{-3}e^{-t} + C. \]

8. Modeling.
Consider a baseball of mass $0.2 \text{ kg}$ thrown directly towards the ground with initial velocity $30 \text{ m/s}$ from the top of a very tall building. Assume the force of air resistance is proportional (with proportionality constant $1 \text{ kg} \cdot \text{s}^{-1}$) to the instantaneous velocity.

Write an initial value problem describing its instantaneous velocity, and solve it. Describe what happens as $t \to \infty$.

Using Newton’s Second Law, we know that $F_{\text{net}} = ma(t)$, namely that the sum of all forces acting on an object results in its acceleration. We also recall that $a(t) = v'(t)$. The two forces acting on the ball are the force of gravity, and, acting in opposition, is the force of air resistance. These are given by:
\[ F_g = mg, \quad F_a = -1v, \]
where we have assigned the minus to the force acting away from the center of the earth and $g$ is the acceleration due to gravity. Thus $F_{\text{net}} = mg - v$. Rewriting with Newton, we have: $mg - v = ma = mv'$, and thus we have the initial value problem in $v(t)$:
\[
\begin{cases}
  v' = g - \frac{v}{2} \\
  v(0) = 30.
\end{cases}
\]

This is a separable problem, and we can rewrite everything as follows: $\frac{-2dv}{v - 2g} = dt$, which can be integrated to yield:
\[ \ln|v - 2g| = -t/2 + C \implies |v - 2g| = ce^{-t/2}. \]
Some caution is warranted here, since the sign of $v - 2g$ matters. But, since $v(0) = 30$ is greater than $0.2 \times 9.8$, we see that, here,
\[ v(t) = ce^{-t/2} + 2g. \]
Invoking the initial condition, we see that $c = 30 - 2g$. Thus as $t \to \infty$, the velocity decreases to a “terminal” velocity, which is constant.

Disclaimer: The above modeling is not accurate and the problem is totally contrived. In reality the pertinent ODE takes into account a quadratic viscous term (based on fluid dynamic principles, themselves based on Reynolds number and other physical properties):
\[ v' = mg - \kappa v^2. \]
To compute an actual terminal gravity, we are looking for the point where the two forces will balance out, at which point the object (in free fall) should experience no change in its velocity. This would be a stationary solution, and thus \( v' = 0 \). In this case we’d see that \( v \sim \sqrt{\frac{mg}{\kappa}} \).

9. Second order constant coefficient ODE.

(a) Consider the second order, constant coefficient differential equation in \( x(t) \):

\[
x'' + bx' + 2x = 0. \tag{1}
\]

Find a value for \( b \) so that solutions to this ODE are damped, but not overdamped—i.e., such that solutions \( x(t) \to 0 \) exponentially as \( t \to \infty \) but still have a periodic component.

For this to occur, we need the characteristic polynomial to have solutions with real and imaginary parts. This occurs when the discriminant \( b^2 - 4ac < 0 \), and thus we need \( b^2 - 4(2) < 0 \implies b^2 < 8 \). This occurs, for instance, if \( b = 2 \). (Note that for \( b < 0 \), the solution would not be damped...as you will see below, the solutions would be negatively damped and would GROW to infinity.)

(b) Consider the second order, constant coefficient differential equation in \( x(t) \):

\[
x'' - 4x' + 4x = g(t). \tag{2}
\]

i. Let \( g(t) = 2e^t \). What is the general solution to (2) in this (inhomogeneous) situation?

The characteristic polynomial is \( r^2 - 4r + 4 = (r - 2)^2 \). Thus we have one real, repeated root: \( r = 2 \). The general solution is

\[ x_g(t) = c_1 e^{2t} + c_2 t e^{2t}. \]

To find the particular solution, we use the method of undetermined coefficients and we guess that the solution has the form \( Ae^t \). Plugging this in, we note that \( (A - 4A + 4A)e^t = 2e^t \), and thus we conclude that \( A = 2 \), and the particular solution is \( x_p(t) = 2e^t \).

Thus the general solution to the inhomogeneous problem is \( x(t) = x_p(t) + x_g(t) = 2e^t + c_1 e^{2t} + c_2 t e^{2t} \).

ii. Let \( g(t) = 0 \) and \( x(0) = 1, \ x'(0) = 0 \).

This is just the homogeneous case, and thus we’ll work with \( x_g(t) = c_1 e^{2t} + c_2 t e^{2t} \).

We need to compute \( x_g' \) to use the second initial condition:

\[ x_g'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} = (2c_1 + c_2)e^{2t} + 2c_2 t e^{2t}. \]

We impose the initial conditions now: first, \( 1 = x_g(0) = c_1 \); secondly, \( 0 = x_g'(0) = 2c_1 + c_2 \). With \( c_1 = 1 \), we conclude that \( c_2 = -2 \). Thus the solution to the IVP is \( x(t) = e^{2t} - 2te^{2t} \).

In this (homogeneous) case what is the solution satisfying (2) and the given initial conditions?
iii. Let \( g(t) = \sin(t) + e^{2t} \).

What is the appropriate form for the particular solution to (2) in this case? (Do not solve for undetermined coefficients.)

Since there is a sinusoidal component, we know the guess must include \( A \sin(t) + B \cos(t) \). In addition, we’d be inclined to guess \( e^{2t} \), however, this is included in the fundamental set for this problem. Thus we must adjust the guess with the multiplicative factor, including a second order term \( Ct^2e^{2t} \). Thus the structure of our guess is

\[
x_p(t) = A \sin(t) + B \cos(t) + Ct^2e^{2t}.
\]

(c) Using the forced ODE

\[
x'' + 9x = \sin(\omega t),
\]

explain in about three sentences the resonance phenomenon.

The fundamental set here is purely sinusoidal, and the general solution to the homogeneous equation is \( x_g(t) = c_1 \sin(3t) + c_2 \cos(3t) \). If we include a periodic forcing, the particular solution to the inhomogeneous problem will include a term of the form \( x_p(t) = A \sin(\omega t) + B \cos(\omega t) \). However, if \( \omega = 3 \), which is to say that the periodic forcing matches the natural frequency of the system, the particular solution needs to be adjusted (as in the previous problem). In this case, we would have \( x_p(t) = t[A \sin(3t) + B \cos(3t)] \). Note that the solution will remain bounded in the case where \( \omega \neq 3 \), but when \( \omega = 3 \), the solution grows unboundedly in time. This is resonance—when a solution is forced near its natural frequency, the solutions grow unboundedly in time.